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Regularity of fractional analogue of k-Hessian operators and a non-local one-phase free boundary problem

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Regularity of fractional analogue of k-Hessian operators and a non-local one-phase free boundary problem

by

Yijing Wu

DISSERTATION

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF TEXAS AT AUSTIN

May 2019

Acknowledgments

I would like to thank my supervisor, Professor Luis Caffarelli, for his support, encouragement, and patience over the course of my graduate studies. Without his guidance and help, I would not have been able to complete this work.

I am thankful to my dissertation committee members Professor Alessio Figalli, Professor Stefania Patrizi and Professor Alexis Vasseur for their help and advice. I would like to thank the mathematics faculty and graduate students at the University of Texas at Austin, as well as all staff and members for their continued support.

I am greatly indebted to my family: my parents and husband for their unconditional encouragement and love.

This work is supported by National Science Foundation grant DMS-1500871.

Regularity of fractional analogue of k-Hessian operators and a non-local one-phase free boundary problem

Publication No. _______________

Yijing Wu, Ph.D.
The University of Texas at Austin, 2019

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We study the regularity theory of fractional analogue of k-Hessian operators. We define the fractional k-Hessian operators as concave envelopes of linear fractional order operators. We have $C^{1,1}$ regularity of viscosity solutions under the set-up of global solutions prescribing data at infinity and global barriers. Then we apply Evans-Krylov theorem to improve the regularity of fractional 2-Hessian operator to $C^{2s+\alpha}$, and the key estimate is to prove the operator is strictly elliptic.

We also study the minimizers of the energy

$$J_\gamma(u) = \int y^{1-2s}|\nabla u|^2 dxdy + \int_{\{y=0\}} u^\gamma dx$$

where $x \in \mathbb{R}^n$ and $y \geq 0$ with $0 < s, \gamma < 1$. This non-local one-phase free boundary problem is an intermediate case of thin obstacle and fractional cavitation problem. We prove the homogeneity of the blow-up profiles and the regularity of free boundary under the flatness condition.

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Chapter 1

Fractional analogue of k-Hessian operators

1.1 Introduction

The Monge-Ampère operator is a special case of k-Hessian operators, which are defined by

$$f_k(D^2u)(x) = (\sum_{1\le i_1<i_2<...<i_k\le n} \lambda_{i_1}\lambda_{i_2}...\lambda_{i_k})^{1/k},$$

for k integer and $1 \le k \le n$. Here $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the matrix $D^2u(x)$, and f_k is concave and elliptic [9] [10] as a function of λ when

$$\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \overline{\Gamma_k}.$$

We denote by Γ_k an open symmetric convex cone defined by

$$\Gamma_k = \{\lambda \in \mathbb{R}^n, S_l(\lambda) > 0, l = 1, 2, ..., k\}.$$

Here

$$S_k(\lambda) = \sum_{1\le i_1<i_2<...<i_k\le n} \lambda_{i_1}\lambda_{i_2}...\lambda_{i_k}$$

is the k-th elementary symmetric polynomial. When $k = n$, Γ_n is the positive cone

$$\Gamma_n = \{\lambda \in \mathbb{R}^n, \lambda_i > 0, i = 1, 2, ..., n\}.$$

We denote by Γ_k the set of symmetric matrices with eigenvalues in Γ_k. In addition, we can also define

$$S_k(B) = \sum_{\alpha \in \{n_k\}} [B]_{\alpha,\alpha}$$

where $\{n_k\}$ denotes the collection of all subsets of k elements from the set {1,2,...,n}. We denote by α any of such subsets and $[B]_{\alpha,\alpha}$ denotes the determinant of the $k \times k$ matrix that results from deleting all rows and columns whose indices are not in α. And

$$f_k(B) = (S_k(B))^{1/k}$$

is homogeneous of degree 1.

One main ingredient of the paper [4] is the following:

The Monge-Ampère equation is a concave fully nonlinear equation. If u is a convex solution solving

$$(\det D^2u)^{1/n}(x) = g(x),$$

then the equation is equivalent to

$$\inf_{M \in \mathcal{M}} L_M u(x) = g(x),$$

where L_M is a linear operator defined by

$$L_M u(x) = trace(MD^2u(x)) = \Delta(u \circ \sqrt{M})(\sqrt{M}^{-1}x),$$

and the set $\mathcal{M}$ consists of all positive symmetric matrices with determinant n^{-n}, independent of x. Moreover, the infimum is realized when M is a constant multiple of the matrix of cofactors of $D^2u(x)$.

Then the fractional analogue of Monge-Ampère equation can be defined as

$$F_s[u](x) = \inf_{M \in \mathcal{M}} \{-C_{n,s}^{-1}(-\Delta)^s(u \circ \sqrt{M})(\sqrt{M}^{-1}x)\},$$

with constant

$$C_{n,s} = \frac{4^s\Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|}.$$

Under this setting, regularity results for fractional Monge-Ampère equation are discussed in [4].

Therefore, it is natural to consider k-Hessian operators as concave envelopes of linear operators. As an analogue of definition of the Monge-Ampère operator, we define

$$\begin{aligned} f_k(D^2u(x)) &= \inf_{M \in \mathcal{M}_k} \{trace(MD^2u(x))\} \\ &= \inf_{M \in \mathcal{M}_k} \{\Delta(u \circ \sqrt{M})(\sqrt{M}^{-1}x))\} \end{aligned}$$

The set $\mathcal{M}_k = Df_k(\Gamma_k)$, where Df_k is the Gâteaux derivative of f_k. A matrix $M \in \mathcal{M}_k$ if $M_{ij} = Df_k(B)E_{ij}$ with $B \in \Gamma_k$ and E_{ij} is the matrix with the i,jth entry being 1 and all other entries being 0. Details and explanations of the set $\mathcal{M}_k$ are further discussed in section 2, and we have two important properties of Df_k.

Remark 1.1.1. 1. The invariance of f_k by orthonormal matrices implies

$$Df_k(PBP^T)A = Df_k(B)P^TAP.$$

By definition of Df_k,

$$
\begin{aligned}
Df_k(PBP^T)A &= \lim_{\epsilon\to 0}\frac{f_k(PBP^T+\epsilon A)-f_k(PBP^T)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{f_k(P(B+\epsilon P^TAP)P^T)-f_k(B)}{\epsilon}\\
&= \lim_{\epsilon\to 0}\frac{f_k(B+\epsilon P^TAP)-f_k(B)}{\epsilon}\\
&= Df_k(B)P^TAP.
\end{aligned}
$$

2. By the strict concavity of f_k in Γ_k, Df_k is strictly monotone and therefore Df_k is a bijection between Γ_k and $\mathcal{M}_k$ by the inverse function theorem.

Then we are able to give a similar definition for fractional analogues of k-Hessian operators:

Definition 1.1.2. Define fractional k-Hessian operators as

$$
\begin{aligned}
F_{k,s}[u](x) &= \inf_{M\in\mathcal{M}_k}\{-C_{n,s}^{-1}(-\Delta)^s(u\circ\sqrt{M})((\sqrt{M}^{-1}x)\}\\
&= \inf_{M\in\mathcal{M}_k}\{P.V.\int_{\mathbb{R}^n}\frac{u(x+y)-u(x)}{|\sqrt{M}^{-1}y|^{n+2s}}\det\sqrt{M}^{-1}dy\}\\
&= \inf_{M\in\mathcal{M}_k}\{\frac{1}{2}\int_{\mathbb{R}^n}\frac{\delta(u,x,y)}{|\sqrt{M}^{-1}y|^{n+2s}}\det\sqrt{M}^{-1}dy\},
\end{aligned}
$$

where

$$\delta(u,x,y)=u(x+y)-2u(x)+u(x-y),$$

and

$$C_{n,s}=\frac{4^s\Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|}.$$

The main idea of this article is to reproduce the regularity results of fractional Monge-Ampère equation in [4] to fractional k-Hessian equations.

In this article, our main purpose is to follow the ideas and set up of the paper [4], and to prove:

(a) On each $n-1$ dimensional space, the fractional Laplacian is bounded from above and strictly positive. (Proposition 1.4.1)

(b) When $k = 2$, the operators that are close to the infimum remain strictly elliptic. (Theorem 1.1.4)

Here we define the strictly elliptic operator:

Definition 1.1.3. For $\epsilon_0 > 0$, we define a non-degenerate and strictly elliptic operator

$$
\begin{aligned}
F_{k,s}^{\epsilon_0}[u](x) &= \inf_{M \in \mathcal{M}_k} \{P.V. \int_{\mathbb{R}^n} \frac{u(x+y) - u(x)}{|\sqrt{M}^{-1} y|^{n+2s}} \det \sqrt{M}^{-1} dy, \lambda_{min}(M) \geq \epsilon_0\} \\
&= \inf_{M \in \mathcal{M}_k} \{\frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u,x,y)}{|\sqrt{M}^{-1} y|^{n+2s}} \det \sqrt{M}^{-1} dy, \lambda_{min}(M) \geq \epsilon_0\}.
\end{aligned}
$$

The main theorem states that the infimum in the definition of $F_{2,s}[u]$ cannot be realized by matrices that are too degenerate, which proves that the fractional analogue of 2-Hessian operators are locally uniformly elliptic.

Theorem 1.1.4. *Consider $1/2 < s < 1$, and assume u is Lipschitz continuous and semi-concave with constants L and SC respectively. If u satisfies*

$$
(1-s)F_{2,s}[u](x) \geq \eta_0 \tag{1.1.1}
$$

for any $x \in \Omega$, in the viscosity sense for some constant $\eta_0 > 0$, then

$$
F_{2,s}[u](x) = F_{2,s}^{\epsilon_0}[u](x) \tag{1.1.2}
$$

for any $x \in \Omega$ in the classical sense, with

$$\epsilon_0 = \epsilon_0(\eta_0, n, s, L, SC) > 0$$

given by (1.4.6).

Remark 1.1.5. We can check in the proofs that ϵ_0 given by (1.4.6) does not converge to 0 as $s \to 1$, and this shows that Theorem 1.1.4 is stable as $s \to 1$.

Remark 1.1.6. Note for the sequel that since u is semi-concave, Lemma 2.2 in paper [4] implies that $F_{2,s}(x)$ is defined in the classical sense for all $x \in \Omega$ and (1.1.1) holds pointwise. For simplicity, we assume that $0 \in \Omega$ and then prove (1.1.2) for $x = 0$.

Under a framework of global solutions prescribing data at infinity and global barriers, which are set up to avoid complexity of dealing with issues from the boundary data for non-local equations, the following results for fractional Monge-Ampère equations also work for fractional k-Hessian equations:

(c) Existence of solutions. (Theorem 1.1.7)

(d) Semi-concavity and Lipschitz continuity of solutions. (Theorem 1.1.8)

(e) The non-local fully nonlinear theory developed in [9] [10] applies, in particular the nonlocal Evans-Krylov theorem.

Theorem 1.1.7. *Assume ϕ is semi-concave and Lipschitz continuous, then there exists a unique viscosity solution of*

$$\begin{cases} F_{k,s}[u](x) = u(x) - \phi(x) \quad \text{in} \ \ \mathbb{R}^n \\ (u - \phi)(x) \to 0 \quad \text{as} \ \ |x| \to \infty. \end{cases}$$

Theorem 1.1.8. *Assume ϕ is semi-concave and Lipschitz continuous, and let v be the viscosity solution of*

$$\begin{cases} F_{k,s}[v](x) = v(x) - \phi(x) \quad in \ \ \mathbb{R}^n \\ (v-\phi)(x) \to 0 \quad as \ \ |x| \to \infty. \end{cases}$$

Then, v is Lipschitz continuous and semi-concave with the same constants as ϕ.

ϕ is the prescribed boundary data at infinity, and acts as a smooth lower barrier function. Following is the requirement on ϕ.
$\phi \in C^{2,\alpha}(\mathbb{R}^n)$ is strictly convex in compact sets and $\phi = \Gamma + \eta$ near infinity, with $\Gamma(x)$ a cone and

$$|\eta(x)| < a|x|^{-\epsilon}, |\nabla\eta(x)| < a|x|^{-1-\epsilon}, \text{and} \ \ |D^2\eta(x)| < a|x|^{-2-\epsilon}$$

for some constants $a > 0$ and $0 < \epsilon < n$. In particular, as $|x| \to \infty$,

$$-(-\Delta)^s\eta(x) = O(|x|^{-2s-\epsilon}),$$

(see (1.2.1) for the definition of the fractional Laplacian) and

$$c_1|x|^{1-2s} \leq -(-\Delta)^s\Gamma(x) \leq c_2|x|^{1-2s}$$

from the homogeneity, where c_1, c_2 are some positive constants depending on the strict convexity of the section of Γ. We normalize ϕ so that $\phi(0) = 0$ and $\nabla\phi(0) = 0$.

The difference between fractional Monge-Ampère operators and k-Hessian operators is the set of matrices M among which we take infimum of fractional linear operators. In Monge-Ampère, we consider the infimum among all positive symmetric matrices with determinant n^{-n}, and in k-Hessian, we consider the infimum among all positive symmetric matrices in the set $\mathcal{M}_k$ (which is discussed in Section 2, Proposition 1.3.2). Hence, we can apply the exact same proofs of existence and $C^{1,1}$ regularity in the fractional Monge-Ampère case, which are carefully explained in section 4, 5 and 6 in [4], to prove Theorem 1.1.7 and Theorem 1.1.8 for our fractional k-Hessian equations.

Thus by what we have proved in (b), that such operators are strictly elliptic, and $C^{1,1}$ estimates in (d), we can apply nonlocal Evans-Krylov theorem [9] [10] to prove solutions of fractional 2-Hessian equations are $C^{2s+\alpha}$, and further classical, under the framework of global solutions prescribing data at infinity and global barriers.

Remark 1.1.9. The proof for strict ellipticity of the operator is required to improve the $C^{1,1}$ regularity to $C^{2s+\alpha}$ regularity. Therefore, we only care about the case $1/2 < s < 1$ in Theorem 1.1.4, or there is no improvement in the regularity. We also care the case as $s \to 1$, and in the Remark 1.1.5, we can see that Theorem 1.1.4 is stable as $s \to 1$.

1.2 Notations and Preliminaries

Given a function u, we denote the second-order increment of u at x in the direction of y as

$$\delta(u, x, y) = u(x + y) + u(x - y) - 2u(x),$$

and fractional Laplacian is defined as

$$\begin{aligned} -(-\Delta)^s u(x) &= C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|x - y|^{n+2s}} dy \\ &= \frac{C_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|x - y|^{n+2s}} dy. \end{aligned} \tag{1.2.1}$$

The constant $C_{n,s}$ is a normalization constant

$$C_{n,s} = \frac{4^s \Gamma(n/2 + s)}{\pi^{n/2} |\Gamma(-s)|}.$$

For square matrices, $A > 0$ means positive definite and $A \geq 0$ positive semidefinite. We denote $\lambda_i(A)$ the eigenvalues of A, in particular $\lambda_{min}(A)$ and $\lambda_{max}(A)$ are the smallest and largest eigenvalues, respectively.

We denote the n-dimensional ball of radius r and center x by $B_r^n(x) = \{y \in \mathbb{R}^n, |y - x| < r\}$, and the corresponding $(n - 1)$-dimensional sphere by $\partial B_r^n(x) = \{y \in \mathbb{R}^n, |y - x| = r\}$. We denote the measure of a $(n - 1)$-dimensional sphere by $|\partial B_r^n|$.

Let $A \subset \mathbb{R}^n$ be an open set. We say that a function $u : A \to \mathbb{R}$ is semi-concave if it is continuous in A and there exists a constant $SC \geq 0$ such that $\delta(u, x, y) \leq SC|y|^2$ for all $x, y \in \mathbb{R}^n$ such that the segment $[x - y, x + y] \subset A$.

The constant SC is called a semi-concavity constant for u in A. Alternatively, a function u is semi-concave in A with constant SC if $u(x) - \frac{SC}{2}|x|^2$ is concave in A. Geometrically, this means that the graph of u can be touched from above at every point by a paraboloid of the type $a + < b, x > + \frac{SC}{2}|x|^2$.

1.3 A representation of local k-Hessian operators and the definition of their non-local counterparts

In this section, we discuss one important representation of Monge-Ampère operator (Proposition 1.3.1). Next we derive a similar representation for local k-Hessian operator (Proposition 1.3.2), show how we construct the set $\mathcal{M}_k$, and give the definition of fractional k-Hessian operator.

We can write the Monge-Ampère operator as a concave envelope of linear operators.

Proposition 1.3.1. *If u is convex, then the Monge-Ampère operator $f(D^2u) = (\det D^2u)^{1/n}$ can be expressed as*

$$f(D^2u) = (\det D^2u)^{1/n} = \inf_{M \in \mathcal{M}} L_M u,$$

where $\mathcal{M}$ is the set of all positive symmetric matrices with determinant n^{-n}, and the linear operator $L_M u$ is defined by

$$L_M u(x) = trace(MD^2u(x)) = \Delta(u \circ \sqrt{M})(\sqrt{M}^{-1}x).$$

Proof of Proposition 1.3.1. Let $A = D^2u(x)$ be a positive matrix, and we consider the Monge-Ampère operator $f(A) = (\det A)^{1/n}$ as a concave envelope of linear operators, that is

$$f(A) = \inf_{B \in \Gamma_n} \{Df(B)(A - B) + f(B)\},$$

and $Df(B) : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the Gâteaux derivative of f, defined by

$$Df(B)A = \lim_{\epsilon\to 0} \frac{f(B + \epsilon A) - f(B)}{\epsilon}.$$

Since f is homogeneous of degree 1, for any $t > 0$,

$$f(tB) = tf(B),$$

and we can prove

$$Df(B)B = \lim_{\epsilon\to 0} \frac{f(B + \epsilon B) - f(B)}{\epsilon} = f(B).$$

Letting $E_{ij} \in \mathbb{R}^{n\times n}$ be the matrix with the i, jth entry being 1 and all other entries being 0, we can calculate

$$det(B + \epsilon E_{ij}) = (b_{ij} + \epsilon)b^*_{ij} + \sum_{k\neq i} b_{kj}b^*_{kj} = detB + \epsilon b^*_{ij}, \tag{1.3.1}$$

where b^*_{ij} is the i, jth entry of the cofactor matrix of B, and thus

$$Df(B)E_{ij} = \lim_{\epsilon\to 0} \frac{(detB + \epsilon b^*_{ij})^{1/n} - (detB)^{1/n}}{\epsilon} = \frac{1}{n}(\det B)^{\frac{1}{n}-1}b^*_{ij},$$

Thus, by linearity,

$$Df(B)A = Df(B)(a_{ij}E_{ij}) = a_{ij}(\frac{1}{n}(\det B)^{\frac{1}{n}-1}b^*_{ij}) = trace(AM^T),$$

where

$$M = Df(B) = \frac{1}{n}(\det B)^{\frac{1}{n}-1}b^*_{ij}.$$

By the property of cofactor matrix B^* that $B^{-1} = (\det B)^{-1} B^*$, we know

$$\det M = n^{-n}.$$

Therefore, by the bijection between Γ_n and $\mathcal{M}$, without loss of generality, we can conclude that

$$(\det D^2 u)^{1/n} = \inf_{M \in \mathcal{M}} L_M u = \inf_{M \in \mathcal{M}} trace(M D^2 u),$$

where $\mathcal{M}$ is the set of all positive symmetric matrices with determinant n^{-n}.

□

Monge-Ampère operator is the n-Hessian operator. Thus, we try to find a similar way of representing the concave k-Hessian operator.

Proposition 1.3.2. *If $D^2 u \in \Gamma_k$, then the k-Hessian operator*

$$f_k(D^2 u) = \Big(\sum_{1 \le i_1 < i_2 < \ldots < i_k \le n} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_k}\Big)^{1/k}$$

is a concave envelope of linear operators, that is

$$f_k(D^2 u) = \inf_{M \in \mathcal{M}_k} \{trace(M D^2 u)\}.$$

A matrix $M \in \mathcal{M}_k$ if there exists a matrix $B \in \Gamma_k$, such that $M = Df_k(B)$. The entries of the matrix M satisfy the following conditions:

$$M_{ii} = \frac{1}{k f_k(B)^{k-1}} \sum_{\alpha \in \{n_k\}, i \in \alpha} [B]_{\alpha_i, \alpha_i}. \tag{1.3.2}$$

Here $\{n_k\}$ denotes the collection of all subsets of k elements from the set $\{1,2,...,n\}$. We denote by α any of such subsets and $[B]_{\alpha,\alpha}$ denotes the determinant of the $k \times k$ matrix that results from deleting all rows and columns whose indices are not in α. And α_i is the set removing index i from set α. When $i \neq j$,

$$M_{ij} = \frac{1}{kf_k(B)^{k-1}} \sum_{\alpha \in \{n_k\}, i,j \in \alpha} (-1)^{l_1+l_2} [B]_{\alpha_i, \alpha_j}, \tag{1.3.3}$$

here l_1, l_2 denotes the position of i and j in the set $\alpha = \{j_1, j_2, ..., j_k\}$, which means $i = j_{l_1}$ and $j = j_{l_2}$, and $j_1 < j_2 < ... < j_k$.

Proof of Proposition 1.3.2. We say a matrix $B \in \Gamma_k$ if eigenvalues of B are in Γ_k. Since f_k is a concave function of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \Gamma_k$, with $\lambda_j, j = 1, 2, ..., n$ eigenvalues of matrix A, we can write

$$f_k(A) = \inf_{B \in \Gamma_k} \{Df_k(B)(A - B) + f_k(B)\}.$$

Here the operator $Df_k(B) : \mathbb{R}^{n \times n} \to \mathbb{R}$ is the Gâteaux derivative of f_k defined by

$$Df_k(B)A = \lim_{\epsilon \to 0} \frac{f_k(B + \epsilon A) - f_k(B)}{\epsilon}.$$

Take a basis $\{E_{ij}\}_{i,j=1}^{n}$ of $\mathbb{R}^{n \times n}$, such that E_{ij} is a matrix with i, j th entry being 1, and all other entries being 0. We write the matrix $A = a_{ij}E_{ij}$. Then by linearity,

$$Df_k(B)A = Df_k(B)(a_{ij}E_{ij}) = a_{ij}Df_k(B)E_{ij} = a_{ij} \lim_{\epsilon \to 0} \frac{f_k(B + \epsilon E_{ij}) - f_k(B)}{\epsilon}.$$

Define a matrix $M \in \mathbb{R}^{n\times n}$ by

$$M_{ij} = Df_k(B)E_{ij} = \lim_{\epsilon\to 0} \frac{f_k(B+\epsilon E_{ij}) - f_k(B)}{\epsilon},$$

then

$$Df_k(B)A = a_{ij}M_{ij} = trace(AM^T),$$

and we write $M = Df_k(B)$ for simplicity and $M_{ij} = Df_k(B)E_{ij}$.

We define

$$S_k(B) = \sum_{\alpha\in\{n_k\}} [B]_{\alpha,\alpha},$$

where $\{n_k\}$ denotes the collection of all subsets of k elements from the set {1,2,...,n}. We denote by α any of such subsets and $[B]_{\alpha,\alpha}$ denotes the determinant of the $k\times k$ matrix that results from deleting all rows and columns whose indices are not in α.

We can calculate that

$$S_k(B+\epsilon E_{ii}) = \sum_{\alpha\in\{n_k\},i\in\alpha} [B+\epsilon E_{ii}]_{\alpha,\alpha} + \sum_{\alpha\in\{n_k\},i\notin\alpha} [B]_{\alpha,\alpha}.$$

As calculated in (1.3.1), when $i\in\alpha$,

$$[B+\epsilon E_{ii}]_{\alpha,\alpha} = [B]_{\alpha,\alpha} + \epsilon[B]_{\alpha_i,\alpha_i}$$

where α_i is the set removing index i from α. Then we can calculate

$$\begin{aligned} S_k(B+\epsilon E_{ii}) &= \sum_{\alpha\in\{n_k\},i\in\alpha} [B]_{\alpha,\alpha} + \epsilon \sum_{\alpha\in\{n_k\},i\in\alpha} [B]_{\alpha_i,\alpha_i} + \sum_{\alpha\in\{n_k\},i\notin\alpha} [B]_{\alpha,\alpha} \\ &= S_k(B) + \epsilon \sum_{\alpha\in\{n_k\},i\in\alpha} [B]_{\alpha_i,\alpha_i}, \end{aligned}$$

and thus

$$M_{ii} = Df_k(B)E_{ii} = \lim_{\epsilon \to 0} \frac{S_k^{1/k}(B + \epsilon E_{ii}) - S_k^{1/k}(B)}{\epsilon}$$
$$= \frac{1}{kf_k(B)^{k-1}} \sum_{\alpha \in \{n_k\}, i \in \alpha} [B]_{\alpha_i, \alpha_i}.$$

Similarly, we can calculate when $i \neq j$,

$$S_k(B + \epsilon E_{ij}) = \sum_{\alpha \in \{n_k\}, i,j \in \alpha} [B + \epsilon E_{ij}]_{\alpha,\alpha} + \sum_{\alpha \in \{n_k\}, i \notin \alpha \text{ or } j \notin \alpha} [B]_{\alpha,\alpha},$$

because, as calculated in (1.3.1), when $i, j \in \alpha$,

$$[B + \epsilon E_{ij}]_{\alpha,\alpha} = [B]_{\alpha,\alpha} + (-1)^{l_1 + l_2} \epsilon [B]_{\alpha_i, \alpha_j}$$

where α_i is the set removing index i from α. And l_1, l_2 denotes the position of i and j in the set $\alpha = \{j_1, ..., j_k\}$, with $i = j_{l_1}$ and $j = j_{l_2}$. Then we can calculate

$$S_k(B + \epsilon E_{ij}) = \sum_{\alpha \in \{n_k\}, i,j \in \alpha} [B]_{\alpha,\alpha} + \epsilon \sum_{\alpha \in \{n_k\}, i,j \in \alpha} (-1)^{l_1 + l_2} [B]_{\alpha_i, \alpha_j} + \sum_{\alpha \in \{n_k\}, i \notin \alpha \text{ or } j \notin \alpha} [B]_{\alpha,\alpha}$$
$$= S_k(B) + \epsilon \sum_{\alpha \in \{n_k\}, i,j \in \alpha} (-1)^{l_1 + l_2} [B]_{\alpha_i, \alpha_j}.$$

Therefore,

$$M_{ij} = Df_k(B)E_{ij} = \lim_{\epsilon \to 0} \frac{S_k^{1/k}(B + \epsilon E_{ij}) - S_k^{1/k}(B)}{\epsilon}$$
$$= \frac{1}{kf_k(B)^{k-1}} \sum_{\alpha \in \{n_k\}, i,j \in \alpha} (-1)^{l_1 + l_2} [B]_{\alpha_i, \alpha_j}.$$

We write $M = Df_k(B)$ to represent $M_{ij} = Df_k(B)E_{ij}$. Then for any matrix $A \in \mathbb{R}^{n \times n}$, $A = a_{ij}E_{ij}$, by linearity,

$$Df_k(B)A = a_{ij} Df_k(B) E_{ij} = a_{ij} M_{ij} = trace(AM^T).$$

Moreover, since f_k is homogeneous of degree 1,

$$Df_k(B)B = \lim_{\epsilon \to 0} \frac{f_k(B + \epsilon B) - f_k(B)}{\epsilon} = \lim_{\epsilon \to 0} \frac{(1+\epsilon) f_k(B) - f_k(B)}{\epsilon} = f_k(B).$$

Therefore,

$$\begin{aligned} f_k(A) &= \inf_{B \in \Gamma_k} \{Df_k(B)(A - B) + f_k(B)\} \\ &= \inf_{B \in \Gamma_k} \{trace(AM^T), M = Df_k(B)\} \\ &= \inf_{M \in \mathcal{M}_k} \{trace(AM^T)\}. \end{aligned}$$

We can write the set

$$\mathcal{M}_k = Df_k(\Gamma_k) = \{M \in \mathbb{R}^{n \times n}, \text{ exist } B \in \Gamma_k, M = Df_k(B)\}.$$

Df_k is a bijection between Γ_k and $\mathcal{M}_k$ by strict concavity of f_k on Γ_k and inverse function theorem.

$\square$

In particular, if $B = diag\{\sigma_1, \sigma_2, ..., \sigma_n\} \in \Gamma_k$ and $f_k(B) = 1$, then

$$M = Df_k(B) = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$$

with

$$\lambda_i = \frac{1}{k}\Big(\sum_{1 \le i_1 < ... < i_{k-1} \le n, i_j \ne i} \sigma_{i_1}\sigma_{i_2}...\sigma_{i_{k-1}}\Big).$$

From Proposition 1.3.2, we write

$$\begin{aligned} f_k(D^2u(x)) &= \inf_{M \in \mathcal{M}_k} \{trace(D^2u(x)M^T)\} \\ &= \inf_{M \in \mathcal{M}_k} \{\Delta(u \circ \sqrt{M})(\sqrt{M}^{-1}x)\}, \end{aligned}$$

and it is natural to give Definition 1.1.2 of fractional k-Hessian operators by writing

$$\begin{aligned} F_{k,s}[u](x) &= \inf_{M \in \mathcal{M}_k} \{-C_{n,s}^{-1}(-\Delta)^s(u \circ \sqrt{M})((\sqrt{M}^{-1}x)\} \\ &= \inf_{M \in \mathcal{M}_k} \{P.V. \int_{\mathbb{R}^n} \frac{u(x+y)-u(x)}{|\sqrt{M}^{-1}y|^{n+2s}} \det \sqrt{M}^{-1} dy\} \\ &= \inf_{M \in \mathcal{M}_k} \{\frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u,x,y)}{|\sqrt{M}^{-1}y|^{n+2s}} \det \sqrt{M}^{-1} dy\}. \end{aligned}$$

Here without loss of generality, we can assume M is symmetric. This follows from the (unique) polar decomposition of $\sqrt{M}^{-1}$, namely $\sqrt{M}^{-1} = OS^{-1}$, where O is orthogonal and S^{-1} is positive definite and symmetric.

1.4 The main results

In this section we prove Theorem 1.1.4, that when $k = 2$, the infimum in the Definition 1.1.2 of $F_{k,s}$, cannot be realized by matrices that are too degenerate, which proves that the fractional 2-Hessian operator is locally strictly elliptic. Then we can apply results for strictly elliptic and concave non-local operators such as Evans-Krylov theorem to the fractional 2-Hessian operators, to get $C^{2s+\alpha}$ estimates for global solutions prescribing data at infinity and global barriers, and further to prove that such solutions are classical.

Our aim is to prove that as $\epsilon \to 0$,

$$\begin{aligned}F^{\epsilon}_{2,s}[u](x) &= \inf_{M\in\mathcal{M}_2}\{-C^{-1}_{n,s}(-\Delta)^s(u\circ\sqrt{M})(\sqrt{M}^{-1}x), \lambda_{min}(M)=\epsilon\}\\ &= \inf_{M\in\mathcal{M}_2}\{\int_{\mathbb{R}^n}\frac{u(x+y)-u(x)}{|\sqrt{M}^{-1}y|^{n+2s}}det\sqrt{M}^{-1}dy, \lambda_{min}(M)=\epsilon\}\\ &\to\infty.\end{aligned}$$

This shows that the infimum cannot be realized by matrices that are too degenerate, which is the result of Theorem 1.1.4. As explained in Remark 1.1.6, for simplicity, we assume that $0\in\Omega$ and then prove (1.1.2) for $x=0$. Let $\lambda_{min}(M)=\epsilon$ and $M\in\mathcal{M}_2$, we try to prove as $\epsilon\to 0$,

$$I=\int_{\mathbb{R}^n}\frac{u(y)-u(0)}{|\sqrt{M}^{-1}y|^{n+2s}}det\sqrt{M}^{-1}dy\to\infty,$$

which means $F^{\epsilon}_{2,s}[u](0)\to\infty$ as $\epsilon\to 0$. To prove this, we want to consider the integral on $\partial B^n_r(0)$ as an average of integrals on $\partial B^{n-1}_r(0)$. Consider a unit vector

$$\tilde{e}(\theta)=(0,0,...,0,\sin\theta,\cos\theta),$$

with $\theta\in(-\pi/2,\pi/2]$. Then

$$span\{\tilde{e}(\theta)\}^{\perp}=span\{\tilde{e}_1,\tilde{e}_2,...,\tilde{e}_{n-1}\}$$

with $\tilde{e}_j, j=1,2,...,n-1$ be the orthonormal basis of the $n-1$ dimensional perpendicular space. Especially, we can consider

$$\tilde{e}_j=(0,0,...,0,1,0,...,0)\quad j=1,2,...,n-2,$$

with 1 on the position j and 0 otherwise, and

$$\tilde{e}_{n-1}=(0,0,...,0,\cos\theta,-\sin\theta).$$

Then for any $y \in \partial B_r^n(0)$, and $y \perp \tilde{e}(\theta)$, we can write $y = (y_1, y_2, ..., y_n)$ as

$$y = x_1\tilde{e}_1 + x_2\tilde{e}_2 + ... + x_{n-1}\tilde{e}_{n-1},$$

and therefore,

$$y_j = x_j, \quad j = 1, 2, ..., n-2,$$

$$y_{n-1} = x_{n-1}\cos\theta,$$

$$y_n = -x_{n-1}\sin\theta.$$

Now let $M \in \mathcal{M}_2$, $M = diag\{\eta_1, \eta_2, ..., \eta_n\}$, and $\sqrt{M}^{-1} = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$. Assume

$$\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = \epsilon^{-1/2},$$

and write the integral in $\mathbb{R}^n$ as an average of $(n-1)$-dimensional subspaces perpendicular to $\tilde{e}(\theta)$, $-\pi/2 < \theta \leq \pi/2$, that is

$$\begin{aligned}
I &= \prod_{j=1}^{n}\lambda_j \int_{\mathbb{R}^n} \frac{u(y)-u(0)}{(\lambda_1^2 y_1^2 + ... + \lambda_n^2 y_n^2)^{\frac{n+2s}{2}}} dy \\
&= \prod_{j=1}^{n}\lambda_j \int_{-\pi/2}^{\pi/2}\int_0^\infty \int_{x\in\partial B_1^{n-1}(0), x\perp\tilde{e}(\theta)} \\
&\frac{u(r(x_1\tilde{e}_1 + ... + x_{n-1}\tilde{e}_{n-1})) - u(0)}{r^{1+2s}(\lambda_1^2 x_1^2 + ... + (\lambda_{n-1}^2\cos^2\theta + \lambda_n^2\sin^2\theta)x_{n-1}^2)^{\frac{n+2s}{2}}} dx dr d\theta \\
&= \prod_{j=1}^{n}\lambda_j \int_{-\theta_0}^{\theta_0} ... d\theta + \prod_{j=1}^{n}\lambda_j \int_{\theta_0 < |\theta| \leq \pi/2} ... d\theta \\
&= I_1 + I_2.
\end{aligned}$$

The second equality in the formula is ensured by formula (3.2) in page 81 of [14]. Here we let $\theta_0 = C\frac{\lambda_{n-1}}{\lambda_n} \leq \sqrt{8}C\epsilon$ which is proved in (1.4.19) based on the structure of $\mathcal{M}_2$, and the constant C depends on $\frac{\mu_1}{\mu_0}$, determined by (1.4.14).

Our aim is to show that as $\epsilon \to 0$, $I_1 \geq C\eta_1^s$ (Proposition 1.4.2), and $I_2 \geq -C\eta_1^{s-1/2}$ (Proposition 1.4.3). With $\eta_1 \geq \frac{1}{4\epsilon}$ proved in (1.4.17), it leads to $I \to \infty$ as $\epsilon \to 0$.

We need to prove the fractional Laplacian of the restriction of u to any $(n-1)$-dimensional subspace is positive and bounded from above and from below:

Proposition 1.4.1. *Assume that u satisfies all conditions in Theorem 1.1.4, then*

$$0 < \mu_0 \leq (1-s)\int_{\mathbb{R}^{n-1}} \frac{u(z_1e_1 + z_2e_2 + ... + z_{n-1}e_{n-1}) - u(0)}{|\bar{z}|^{n-1+2s}} d\bar{z} \leq \mu_1.$$

for every choice $\{e_j\}_{j=1}^{n-1}$ of $n-1$ orthonormal vectors of $\mathbb{R}^n$, where

$$\mu_0 = \mu_0(\eta_0, n, s, L, SC)$$

is given by (1.4.12), *and*

$$\mu_1 = \mu_1(n, s, L, SC)$$

is given by (1.4.13).

Proposition 1.4.2. *Assume that u satisfies all conditions in Theorem 1.1.4. Let $M \in \mathcal{M}_2$, and $M = diag\{\eta_1, \eta_2, ..., \eta_n\}$, $\sqrt{M}^{-1} = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$,* *with* $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = \epsilon^{-1/2}$. *Let* $\theta_0 = C\frac{\lambda_{n-1}}{\lambda_n} \leq \sqrt{8}C\epsilon$ *which is proved in* (1.4.19) *based on the structure of* $\mathcal{M}_2$*, with the constant* C *determined by* (1.4.14)*, the integral*

$$I_1 = \prod_{j=1}^{n} \lambda_j \int_{-\theta_0}^{\theta_0} \int_0^{\infty} \int_{x \in \partial B_1^{n-1}(0), x \perp \tilde{e}(\theta)}$$
$$\frac{u(r(x_1\tilde{e}_1 + ... + x_{n-1}\tilde{e}_{n-1})) - u(0)}{r^{1+2s}(\lambda_1^2 x_1^2 + ... + (\lambda_{n-1}^2 \cos^2\theta + \lambda_n^2 \sin^2\theta)x_{n-1}^2)^{\frac{n+2s}{2}}} dx dr d\theta$$
$$\geq \frac{C_4}{1-s}\eta_1^s.$$

Here $C_4 = C_4(n, s, \eta_0, L, SC)$ *is given by* (1.4.21)*, and* $\eta_1 \geq \frac{1}{4\epsilon}$ *is proved by* (1.4.17)*.*

Proposition 1.4.3. *Assume that* u *satisfies all conditions in Theorem 1.1.4. Let* $M \in \mathcal{M}_2$*, and* $M = diag\{\eta_1, \eta_2, ..., \eta_n\}$*,* $\sqrt{M}^{-1} = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$*, with* $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = \epsilon^{-1/2}$*. Let* $\theta_0 = C\frac{\lambda_{n-1}}{\lambda_n} \leq \sqrt{8}C\epsilon$ *which is proved in* (1.4.19) *based on the structure of* $\mathcal{M}_2$*, with the constant* C *determined by* (1.4.14)*. The integral*

$$\prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(y_1, y_2, ..., y_{n-1}, 0) - u(0)}{(\lambda_1^2 y_1^2 + ... + \lambda_{n-1}^2 y_{n-1}^2)^{\frac{n+2s-1}{2}}} d\bar{y} \geq -C\epsilon^{s-1/2},$$

and

$$\prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(y_1, y_2, ..., y_{n-1}, 0) - u(0)}{(\lambda_1^2 y_1^2 + ... + \lambda_n^2 y_{n-1}^2)^{\frac{n+2s-1}{2}}} d\bar{y} \geq -C\eta_1^{s-1/2}.$$

This shows

$$I_2 = \prod_{j=1}^{n} \lambda_j \int_{|\theta| \geq \theta_0} \int_0^{\infty} \int_{x \in \partial B_1^{n-1}(0), x \perp \tilde{e}(\theta)}$$
$$\frac{u(r(x_1\tilde{e}_1 + ... + x_{n-1}\tilde{e}_{n-1})) - u(0)}{r^{1+2s}(\lambda_1^2 x_1^2 + ... + (\lambda_{n-1}^2 \cos^2\theta + \lambda_n^2 \sin^2\theta)x_{n-1}^2)^{\frac{n+2s}{2}}} dx dr d\theta$$
$$\geq -C_8\eta_1^{s-1/2}.$$

Proposition 1.4.2 and Proposition 1.4.3 together prove the main theorem:

Proof of Theorem 1.1.4. Let P be an orthogonal matrix such that

$$P^T \sqrt{M}^{-1} P = J = diag\{\lambda_1, ..., \lambda_n\}.$$

and $M \in \mathcal{M}_2$, with $\lambda_{\min}(M) = \epsilon$. then by Proposition 1.4.2 and Proposition 1.4.3, when

$$\epsilon < \epsilon_1 = (\frac{C_4}{4C_8(1-s)})^2, \tag{1.4.1}$$

we can see

$$\begin{aligned} I &= \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|\sqrt{M}^{-1} y|^{n+2s}} \det \sqrt{M}^{-1} dy \\ &= \prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|\lambda_1^2 y_1^2 + ... + \lambda_n^2 y_n^2|^{(n+2s)/2}} dy \\ &= I_1 + I_2 \\ &\geq \frac{C_4}{1-s} \eta_1^s - C_8 \eta_1^{s-1/2} \\ &\geq \frac{1}{2} \frac{C_4}{1-s} \eta_1^s \\ &\geq \frac{1}{2} \frac{C_4}{1-s} (\frac{1}{4\epsilon})^s, \end{aligned} \tag{1.4.2}$$

with $\eta_1 \geq \frac{1}{4\epsilon}$ proved by (1.4.17).

Also, since the identity matrix $I \in \Gamma_2$,

$$M_0 = Df_2(I) = \sqrt{\frac{n-1}{2n}} I \in \mathcal{M}_2,$$

and we can obtain

$$
\begin{aligned}
F_{2,s}[u](0) &= \inf_{M\in\mathcal{M}_2}\{P.V.\int_{\mathbb{R}^n}\frac{u(y)-u(0)}{|\sqrt{M}^{-1}y|^{n+2s}}\det\sqrt{M}^{-1}dy\}\\
&\le \int_{\mathbb{R}^n}\frac{u(y)-u(0)}{|\sqrt{M_0}^{-1}y|^{n+2s}}\det\sqrt{M_0}^{-1}dy\\
&= (\frac{n-1}{2n})^{s/2}\int_{\mathbb{R}^n}\frac{u(y)-u(0)}{|y|^{n+2s}}dy\\
&\le (\frac{n-1}{2n})^{s/2}\int_{\mathbb{R}^n}\frac{\min\{2L|y|,SC|y|^2\}}{|y|^{n+2s}}dy\\
&\le (\frac{n-1}{2n})^{s/2}(SC\int_{B_1^n}\frac{1}{|y|^{n+2s-2}}dy+2L\int_{\mathbb{R}^n\setminus B_1^n}\frac{1}{|y|^{n+2s-1}}dy)\\
&= (\frac{n-1}{2n})^{s/2}(SC|\partial B_1^n|\int_0^1 r^{1-2s}dr+2L|\partial B_1^n|\int_1^\infty r^{-2s}dr)\\
&\le (\frac{n-1}{2n})^{s/2}\max\{SC,2L\}|\partial B_1^n|(\int_0^1 r^{1-2s}dr+\int_1^\infty r^{-2s}dr)\\
&= (\frac{n-1}{2n})^{s/2}\frac{|\partial B_1^n|}{|\partial B_1^{n-1}|}\frac{\mu_1}{1-s}.
\end{aligned}
\tag{1.4.3}
$$

with μ_1 defined in (1.4.13).

Therefore, when ϵ is small enough, for instance, when

$$\epsilon \le \epsilon_2 = \frac{1}{4}\sqrt{\frac{n}{n-1}}(\frac{C_4|\partial B_1^{n-1}|}{2\mu_1|\partial B_1^n|})^{1/s}, \tag{1.4.4}$$

we can see

$$\frac{1}{2}\frac{C_4}{1-s}(4\epsilon)^{-s} > (\frac{n-1}{2n})^{s/2}\frac{|\partial B_1^n|}{|\partial B_1^{n-1}|}\frac{\mu_1}{1-s}. \tag{1.4.5}$$

Now we take

$$\epsilon_0 = \min\{\epsilon_1,\epsilon_2\}, \tag{1.4.6}$$

with ϵ_1 defined in (1.4.1) and ϵ_2 defined in (1.4.4). Combining (1.4.2), (1.4.3) and (1.4.5), we can obtain

$$
\begin{aligned}
&\inf_{M\in\mathcal{M}_2}\{\frac{1}{2}\int_{\mathbb{R}^n}\frac{\delta(u,0,y)}{|\sqrt{M}^{-1}y|^{n+2s}}\det\sqrt{M}^{-1}dy, \lambda_{min}(M)\le\epsilon_0\}\\
&\ge\frac{1}{2}\frac{C_4}{1-s}\epsilon_0^{-s}\\
&>(\frac{n-1}{2n})^{s/2}\frac{|\partial B_1^n|}{|\partial B_1^{n-1}|}\frac{\mu_1}{1-s}\\
&\ge F_{2,s}[u](0).
\end{aligned}
$$

Therefore,

$$
\inf_{M\in\mathcal{M}_2}\{\frac{1}{2}\int_{\mathbb{R}^n}\frac{\delta(u,0,y)}{|\sqrt{M}^{-1}y|^{n+2s}}\det\sqrt{M}^{-1}dy, \lambda_{min}(M)\le\epsilon_0\}>F_{2,s}[u](0),
$$

and thus,

$$
F_{2,s}[u](0)=F_{2,s}^{\epsilon_0}[u](0),
$$

with

$$
\epsilon_0=\epsilon_0(n,s,\eta_0,S,LC)
$$

given by (1.4.6). Moreover, we can see ϵ_0 does not converge to 0 as $s\to 1$. This completes the proof for Theorem 1.1.4. □

Now we prove Proposition 1.4.1. Given $\epsilon>0$, find $h(\epsilon)$ such that

$$
B=diag\{\frac{2}{n-1}\epsilon,\frac{2}{n-1}\epsilon,...,\frac{2}{n-1}\epsilon,h(\epsilon)\}\in\Gamma_2,
$$

and

$$
1=S_2(B)=2\epsilon h(\epsilon)+\frac{2(n-2)}{n-1}\epsilon^2.
$$

This means

$$
h(\epsilon)=\frac{1-\frac{2(n-2)}{n-1}\epsilon^2}{2\epsilon}, \tag{1.4.7}
$$

and when ϵ is small enough, $h(\epsilon) \approx \frac{1}{2\epsilon}$. Then as defined, $M(B) = Df_2(B) \in \mathcal{M}_2$ and

$$M(B) = \frac{1}{2S_2(B)^{1/2}} diag\{\frac{2(n-2)}{n-1}\epsilon + h(\epsilon), \frac{2(n-2)}{n-1}\epsilon + h(\epsilon), ..., \frac{2(n-2)}{n-1}\epsilon + h(\epsilon), 2\epsilon\}.$$

So we can write $\sqrt{M}^{-1} = diag\{g(\epsilon), g(\epsilon), ..., g(\epsilon), \epsilon^{-1/2}\}$, where

$$g(\epsilon) = (\frac{n-2}{n-1}\epsilon + \frac{h(\epsilon)}{2})^{-1/2}. \tag{1.4.8}$$

We can see $g(\epsilon) \approx 2\sqrt{\epsilon}$ when ϵ is very small. Then, since $M \in \mathcal{M}_2$, by the equation (1.1.1),

$$\begin{aligned} 0 < \frac{\eta_0}{1-s} &\le \det(\sqrt{M}^{-1}) \int_{\mathbb{R}^n} \frac{u(\bar{y}, y_n) - u(0)}{(g(\epsilon)^2|\bar{y}|^2 + \frac{1}{\epsilon}y_n^2)^{\frac{n+2s}{2}}} dy \\ &= g(\epsilon)^{n-1}\epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{u(\bar{y}, y_n) - u(\bar{y}, 0)}{(g(\epsilon)^2|\bar{y}|^2 + \frac{1}{\epsilon}y_n^2)^{\frac{n+2s}{2}}} dy \\ &\quad + g(\epsilon)^{n-1}\epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{u(\bar{y}, 0) - u(0)}{(g(\epsilon)^2|\bar{y}|^2 + \frac{1}{\epsilon}y_n^2)^{\frac{n+2s}{2}}} dy \\ &= J_1 + J_2. \end{aligned}$$

Lemma 1.4.4 gives an estimate of J_1 by semi-concavity and Lipschitz continuity of u.

Lemma 1.4.4. *Assume that u satisfies all conditions in Theorem 1.1.4. Take*

$$\sqrt{M}^{-1} = diag\{g(\epsilon), g(\epsilon), ..., g(\epsilon), \epsilon^{-1/2}\},$$

with $g(\epsilon)$ defined in (1.4.8)*, then*

$$J_1 = g(\epsilon)^{n-1}\epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{u(\bar{y}, y_n) - u(\bar{y}, 0)}{(g(\epsilon)^2|\bar{y}|^2 + \frac{1}{\epsilon}y_n^2)^{\frac{n+2s}{2}}} dy \le \epsilon^s C_1 C_2,$$

where $C_1 = C_1(s, L, SC)$ and $C_2 = C_2(n, s)$ are given by (1.4.9) *and* (1.4.10) *respectively.*

Proof. By Lipschitz continuity and semi-concavity of u,

$$J_1 \le \frac{1}{2} g(\epsilon)^{n-1} \epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{\min\{2L|y_n|, SC|y_n|^2\}}{(g(\epsilon)^2|\bar{y}|^2 + \frac{1}{\epsilon}y_n^2)^{\frac{n+2s}{2}}} dy,$$

then we can do the change of variables

$$z_n = y_n, z_j = \frac{y_j}{|y_n|}\sqrt{\epsilon} g(\epsilon), j = 1, 2, ..., n-1.$$

Then

$$dz = dy \frac{1}{|y_n|^{n-1}} (\sqrt{\epsilon} g(\epsilon))^{n-1},$$

and

$$\begin{aligned}
J_1 &\le \frac{1}{2} g(\epsilon)^{n-1} \epsilon^{-1/2} (\sqrt{\epsilon} g(\epsilon))^{1-n} \int_{\mathbb{R}^n} \frac{\min\{2L|z_n|, SC|z_n|^2\}}{(1+|\bar{z}|^2)^{\frac{n+2s}{2}} |z_n|^{n+2s-n+1} \epsilon^{-(n+2s)/2}} d\bar{z} dz_n \\
&\le \frac{1}{2} \epsilon^s \int_{\mathbb{R}} \frac{\min\{2L|z_n|, SC|z_n|^2\}}{|z_n|^{1+2s}} dz_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|\bar{z}|^2)^{\frac{n+2s}{2}}} d\bar{z} \\
&\le \epsilon^s C_1 C_2.
\end{aligned}$$

Here we define two constants C_1, C_2 by following:

$$C_1 = C_1(s, L, SC) = \frac{1}{2} \int_{\mathbb{R}} \frac{\min\{2L|z_n|, SC|z_n|^2\}}{|z_n|^{1+2s}} dz_n, \tag{1.4.9}$$

$$C_2 = C_2(n, s) = \int_{\mathbb{R}^{n-1}} \frac{1}{(1+|\bar{z}|^2)^{\frac{n+2s}{2}}} d\bar{z}. \tag{1.4.10}$$

□

Then Lemma 1.4.5 gives an estimate of the integral J_2.

Lemma 1.4.5. *Assume that u satisfies all conditions in Theorem 1.1.4. Take*

$$\sqrt{M}^{-1} = diag\{g(\epsilon), g(\epsilon), ..., g(\epsilon), \epsilon^{-1/2}\},$$

then

$$J_2 = g(\epsilon)^{-2s} C_3 \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z},$$

where $C_3 = C_3(n,s)$ *are given by* (1.4.11).

Proof.

$$J_2 = g(\epsilon)^{n-1} \epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{u(\bar{y},0) - u(0)}{(g(\epsilon)^2 |\bar{y}|^2 + \frac{1}{\epsilon} y_n^2)^{\frac{n+2s}{2}}} dy.$$

Let

$$z_j = y_j, \quad j = 1, 2, ..., n-1,$$

$$z_n = (\sqrt{\epsilon} g(\epsilon))^{-1} \frac{y_n}{|\bar{y}|},$$

and we can get

$$dz = dy (\sqrt{\epsilon} g(\epsilon) |\bar{y}|)^{-1},$$

and

$$\begin{aligned}
J_2 &= g(\epsilon)^{n-1} \epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{u(\bar{y},0) - u(0)}{(g(\epsilon)^2 |\bar{y}|^2 + \frac{1}{\epsilon} y_n^2)^{\frac{n+2s}{2}}} dy \\
&= (\sqrt{\epsilon} g(\epsilon))^{-1} g(\epsilon)^{n-1} \epsilon^{-1/2} \int_{\mathbb{R}^n} \frac{u(\bar{z},0) - u(0)}{g(\epsilon)^{n+2s} |\bar{z}|^{n+2s-1} (1 + z_n^2)^{\frac{n+2s}{2}}} d\bar{z} dz_n \\
&= g(\epsilon)^{-2s} \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \int_{\mathbb{R}} \frac{1}{(1 + z_n^2)^{\frac{n+2s}{2}}} dz_n \\
&= g(\epsilon)^{-2s} C_3 \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z}.
\end{aligned}$$

Here we define a constant C_3 by the following:

$$C_3 = C_3(n,s) = \int_{\mathbb{R}} \frac{1}{(1 + z_n^2)^{\frac{n+2s}{2}}} dz_n. \tag{1.4.11}$$

$\square$

Then combining the estimates for J_1 and J_2, we can prove Proposition 1.4.1:

Proof. From the equation, we can see

$$0 < \frac{\eta_0}{1-s} \leq J_1 + J_2 \leq \epsilon^s C_1 C_2 + g(\epsilon)^{-2s} C_3 \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z},$$

and therefore,

$$\int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \geq \frac{\frac{\eta_0}{1-s} - \epsilon^s C_1 C_2}{C_3 g(\epsilon)^{-2s}}.$$

So we only need to take $\epsilon = \epsilon_3$ small enough such that

$$\eta_0 \geq 2(1-s) C_1 C_2 \epsilon_3^s,$$

that is

$$\epsilon_3 \leq (\frac{\eta_0}{2(1-s) C_1 C_2})^{1/s},$$

then

$$\int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0) - u(0)}{|\bar{z}|^{n+2s-1}} d\bar{z} \geq \frac{\eta_0}{2(1-s) C_3} g(\epsilon_3)^{2s}.$$

We have calculated in (1.4.7) and (1.4.8) that

$$g(\epsilon) = (\frac{1}{4\epsilon} + \frac{n-2}{2(n-1)}\epsilon)^{-1/2},$$

thus

$$g(\epsilon_3)^{2s} = (\frac{1}{4\epsilon_3} + \frac{n-2}{2(n-1)}\epsilon_3)^{-s},$$

and we can define

$$\mu_0 = \mu_0(n, s, \eta_0, L, SC) = \frac{\eta_0}{2C_3} g(\epsilon_3)^{2s}, \tag{1.4.12}$$

and then we obtain the estimates that

$$(1-s)\int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0)-u(0)}{|\bar{z}|^{n+2s-1}}d\bar{z} \geq \mu_0 > 0.$$

By doing an orthonormal transformation, we can show if $\{e_j\}_{j=1}^{n-1}$ are $n-1$ orthonormal vectors of $\mathbb{R}^n$,

$$(1-s)\int_{\mathbb{R}^{n-1}} \frac{u(z_1e_1+z_2e_2+...+z_{n-1}e_{n-1})-u(0)}{|\bar{z}|^{n+2s-1}}d\bar{z} \geq \mu_0 > 0.$$

On the other hand, if u is Lipschitz continuous and semi-concave, then

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0)-u(0)}{|\bar{z}|^{n+2s-1}}d\bar{z} &\leq \int_{\mathbb{R}^{n-1}} \frac{\min\{2L|\bar{z}|, SC|\bar{z}|^2\}}{|\bar{z}|^{n+2s-1}}d\bar{z} \\
&\leq SC\int_{B_1^{n-1}} \frac{1}{|\bar{z}|^{n+2s-3}}d\bar{z} + 2L\int_{\mathbb{R}^{n-1}\backslash B_1^{n-1}} \frac{1}{|\bar{z}|^{n+2s-2}}d\bar{z} \\
&= SC|\partial B_1^{n-1}|\int_0^1 r^{1-2s}dr + 2L|\partial B_1^{n-1}|\int_1^\infty r^{-2s}dr \\
&\leq \max\{SC, 2L\}|\partial B_1^{n-1}|(\int_0^1 r^{1-2s}dr + \int_1^\infty r^{-2s}dr) \\
&= \frac{\mu_1}{1-s}.
\end{aligned}$$

Since $1/2 < s < 1$,

$$\int_0^1 r^{1-2s}dr + \int_1^\infty r^{-2s}dr = \frac{1}{2-2s} + \frac{1}{2s-1}$$

is bounded, and

$$\mu_1 = \mu_1(n,s,L,SC) = (1-s)\max\{SC, 2L\}|\partial B_1^{n-1}|(\frac{1}{2-2s} + \frac{1}{2s-1}). \tag{1.4.13}$$

□

With the estimates in Proposition 1.4.1, now we start to prove Proposition 1.4.2. The main idea is that, when the smallest eigenvalue of matrix M is close to 0, there are some constraints on the other eigenvalues and their square root inverse λ_j, since the matrix is in the set $\mathcal{M}_2$. We prove that $\frac{1}{\lambda_1^{n+2s}} - \frac{1}{\lambda_{n-1}^{n+2s}}$ is very small compared with $\frac{1}{\lambda_1^{n+2s}}$. This and the lower bound in Proposition 1.4.1 prove that the integral on a (n-1)-dimensional subspace, close to $\{x_n = 0\}$, is very large.

Proof of Proposition 1.4.2. Our aim is to show that when ϵ is very small, $I_1 \geq \frac{1}{1-s} C_4 \mu_0 \epsilon^{-s}$. Recall that $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = \sqrt{\epsilon}^{-1}$ with λ_i eigenvalues of $\sqrt{M}^{-1}$.
We take $\theta_0 = C\frac{\lambda_{n-1}}{\lambda_n} \leq \sqrt{8}C\epsilon$ which is proved in (1.4.19) based on the structure of $\mathcal{M}_2$, and the constant C depends on $\frac{\mu_1}{\mu_0}$, determined by (1.4.14). When $|\theta| \leq \theta_0$,

$$\lambda_{n-1}^2 \cos^2\theta + \lambda_n^2 \sin^2\theta \leq (1 + C^2)\lambda_{n-1}^2$$

and thus,

$$\lambda_1^2 \leq \lambda_1^2 x_1^2 + ...(\lambda_{n-1}^2 \cos^2\theta + \lambda_n^2 \sin^2\theta) x_{n-1}^2 \leq (1 + C^2)\lambda_{n-1}^2.$$

Let

$$A(\theta) = \int_0^\infty \int_{\{x \in \partial B_1^{n-1}(0), x \perp \tilde{e}(\theta), u(rx) - u(0) > 0\}} \frac{u(rx) - u(0)}{r^{1+2s}} dxdr \geq 0,$$

and

$$B(\theta) = \int_0^\infty \int_{\{x \in \partial B_1^{n-1}(0), x \perp \tilde{e}(\theta), u(rx) - u(0) \leq 0\}} \frac{u(rx) - u(0)}{r^{1+2s}} dxdr \leq 0.$$

Then by Proposition 1.4.1, for every θ,

$$A(\theta) + B(\theta) \geq \frac{\mu_0}{1-s} > 0,$$

and

$$A(\theta) \leq \frac{\mu_1}{1-s},$$

which leads to

$$A(\theta) \geq \frac{\mu_0}{1-s},$$

and

$$0 \geq B(\theta) \geq \frac{\mu_0 - \mu_1}{1-s}.$$

Then we have the following estimates

$$\begin{aligned}
(1-s)I_1 = (1-s)\prod_{j=1}^{n}\lambda_j \int_{-\theta_0}^{\theta_0}\int_0^{\infty}\int_{x\in\partial B_1^{n-1}(0)} \\
\frac{u(rx)-u(0)}{r^{1+2s}} \frac{1}{(\lambda_1^2 x_1^2 + ...(\lambda_{n-1}^2\cos^2\theta + \lambda_n^2\sin^2\theta)x_{n-1}^2)^{\frac{n+2s}{2}}} dx dr d\theta \\
\geq (1-s)\prod_{j=1}^{n}\lambda_j \int_{-\theta_0}^{\theta_0} (A(\theta)\frac{(1+C^2)^{-(n+2s)/2}}{\lambda_{n-1}^{n+2s}} + B(\theta)\frac{1}{\lambda_1^{n+2s}})d\theta \\
\geq 2\theta_0 \prod_{j=1}^{n}\lambda_j (\frac{\mu_0(1+C^2)^{-(n+2s)/2}}{\lambda_{n-1}^{n+2s}} + \frac{\mu_0-\mu_1}{\lambda_1^{n+2s}}) \\
\geq (2C\lambda_1...\lambda_{n-2}\lambda_{n-1}^2)(\frac{\mu_0}{\lambda_1^{n+2s}} + \mu_1(\frac{(1+C^2)^{-(n+2s)/2}}{\lambda_{n-1}^{n+2s}} - \frac{1}{\lambda_1^{n+2s}})) \\
\geq 2C\lambda_1^n(\frac{\mu_0 + \mu_1(C_5 - 1)}{\lambda_1^{n+2s}} + C_5\mu_1(\frac{1}{\lambda_{n-1}^{n+2s}} - \frac{1}{\lambda_1^{n+2s}})).
\end{aligned}$$

Let

$$C_5 = (1+C^2)^{-(n+2s)/2} < 1$$

and take constant C such that

$$\mu_0 + \mu_1(C_5 - 1) \geq \mu_0/2,$$

i.e., take

$$C = \sqrt{(1 - \frac{\mu_0}{2\mu_1})^{\frac{-2}{n+2s}} - 1} \tag{1.4.14}$$

and

$$C_5 = 1 - \frac{\mu_0}{2\mu_1}. \tag{1.4.15}$$

Now let's see what constraint we have on λ_j when the smallest eigenvalue of matrix $M \in \mathcal{M}_2$ is ϵ. We want to show that the non-negative $\frac{1}{\lambda_1^{n+2s}} - \frac{1}{\lambda_{n-1}^{n+2s}}$ is very small compared with $\frac{1}{\lambda_1^{n+2s}}$.

Let $B = diag\{\sigma_1, \sigma_2, ..., \sigma_n\} \in \Gamma_2$. Assume $\sigma_1 \leq \sigma_2 \leq ... \leq \sigma_n$, and $\sum \sigma_i \sigma_j = 1$. Then $M = diag\{\eta_1, \eta_2, ..., \eta_n\} = Df_2(B)$, with $\eta_1 \geq \eta_2 \geq ... \geq \eta_n = \epsilon$, and

$$\eta_j = \frac{1}{2}(\sum_{i=1}^{n} \sigma_i - \sigma_j).$$

Then

$$\sigma_1 + \sigma_2 + ... + \sigma_{n-1} = 2\epsilon = 2\eta_n.$$

Let $Q = \sigma_2 + \sigma_3 + ... + \sigma_{n-1}$. Since $\sigma_1 \leq \sigma_2 \leq ... \leq \sigma_{n-1}$ and $\sum_{j=1}^{n-1} \sigma_j = 2\epsilon$, which leads to $\sigma_1 \leq \frac{2\epsilon}{n-1}$, thus $Q = 2\epsilon - \sigma_1 \geq \frac{2(n-2)}{n-1}\epsilon$. Since $\sum_{1 \leq i < j \leq n} \sigma_i \sigma_j = 1$, we have

$$\begin{aligned}
1 &= \sigma_n(\sum_{i=1}^{n-1}\sigma_i) + \sum_{1\le i<j\le n-1}\sigma_i\sigma_j \\
&= \sigma_n(2\epsilon) + \sigma_1 Q + \sum_{2\le i<j\le n-1}\sigma_i\sigma_j \\
&\le 2\epsilon\sigma_n + (2\epsilon - Q)Q + \frac{Q^2}{2} \\
&= 2\epsilon\sigma_n + 2\epsilon Q - \frac{Q^2}{2}.
\end{aligned}$$

Then

$$\sigma_n \ge \frac{1 + Q^2/2 - 2\epsilon Q}{2\epsilon}. \tag{1.4.16}$$

Therefore

$$\eta_1 = \frac{1}{2}(Q + \sigma_n) \ge \frac{1 + Q^2/2}{4\epsilon} \ge \frac{1}{4\epsilon}. \tag{1.4.17}$$

In addition, since $\sigma_1 = 2\epsilon - Q$, and $\sigma_{n-1} = 2\epsilon - \sigma_1 - \sigma_2 - ... - \sigma_{n-2} \le 2\epsilon - (n-2)\sigma_1$, so

$$0 \ge \sigma_1 - \sigma_{n-1} \ge (2n-4)\epsilon - (n-1)Q,$$

and this means

$$\eta_{n-1} - \eta_1 \ge \frac{1}{2}((2n-4)\epsilon - (n-1)Q).$$

We can see when $\epsilon \le (4(n-1)(n-3))^{-1/2}$,

$$\begin{aligned}
\eta_{n-1} &\ge (n-2)\epsilon - \frac{n-1}{2}Q + \frac{1 + Q^2/2}{4\epsilon} \\
&= \frac{1 + \frac{1}{2}(Q - 2(n-1)\epsilon)^2 - (2n^2 - 8n + 6)\epsilon^2}{4\epsilon} \qquad (1.4.18) \\
&\ge \frac{1}{8\epsilon}.
\end{aligned}$$

Thus by $\lambda_j = \eta_j^{-1/2}$, and $\lambda_n = \eta_n^{-1/2} = \epsilon^{-1/2}$,

$$\frac{\lambda_{n-1}}{\lambda_n} \leq \frac{\sqrt{8\epsilon}}{\sqrt{\epsilon}^{-1}} \leq \sqrt{8}\epsilon \tag{1.4.19}$$

Thus

$$0 \leq \frac{\eta_1 - \eta_{n-1}}{\eta_1} \leq 2\epsilon \frac{(n-1)Q - (2n-4)\epsilon}{1 + Q^2/2} \leq C\epsilon \to 0. \tag{1.4.20}$$

We can also calculate

$$\begin{aligned} \frac{1}{\lambda_{n-1}^{n+2s}} - \frac{1}{\lambda_1^{n+2s}} &\geq \frac{n+2s}{2} \frac{1}{\lambda_1^{n+2s-2}} \left(\frac{1}{\lambda_{n-1}^2} - \frac{1}{\lambda_1^2}\right) \\ &\geq \frac{n+2s}{4} \frac{1}{\lambda_1^{n+2s-2}} ((2n-4)\epsilon - (n-1)Q). \end{aligned}$$

Therefore,

$$\begin{aligned} (1-s)I_1 &\geq 2C\lambda_1^n \left(\frac{\mu_0 + \mu_1(C_5 - 1)}{\lambda_1^{n+2s}} + C_5\mu_1\left(\frac{1}{\lambda_{n-1}^{n+2s}} - \frac{1}{\lambda_1^{n+2s}}\right)\right) \\ &\geq \frac{C\mu_0}{2}\eta_1^s + \frac{C\mu_0}{2}\eta_1^s + CC_5\mu_1(n+2s)\eta_1^{s-1}\frac{1}{2}((2n-4)\epsilon - (n-1)Q) \\ &\geq \frac{C\mu_0}{2}\eta_1^s + \eta_1^{s-1}\left(\frac{C\mu_0}{2}\eta_1 + C_6\epsilon - C_7 Q\right) \\ &\geq \frac{C\mu_0}{2}\eta_1^s + \eta_1^{s-1}\left(\frac{C\mu_0}{2}\frac{1 + Q^2/2}{4\epsilon} + C_6\epsilon - C_7 Q\right) \\ &\geq \frac{C\mu_0}{2}\eta_1^s + \eta_1^{s-1}\left(\frac{C\mu_0}{8\epsilon} + \left(\sqrt{\frac{C\mu_0}{16\epsilon}}Q - C_7\sqrt{\frac{4\epsilon}{C\mu_0}}\right)^2 + C_6\epsilon - C_7^2\frac{4\epsilon}{C\mu_0}\right) \\ &\geq \frac{C\mu_0}{2}\eta_1^s + 0 \\ &\geq C_4\eta_1^s, \end{aligned}$$

with

$$C_4 = C_4(n, s, L, SC, \eta_0) = \frac{C\mu_0}{2} = \frac{\mu_0}{2}\sqrt{(1 - \frac{\mu_0}{2\mu_1})^{\frac{-2}{n+2s}} - 1}. \tag{1.4.21}$$

$\square$

Then we want to prove the Proposition 1.4.3.

Proof of Proposition 1.4.3. Take a matrix $\Sigma = diag\{\sigma_1, ..., \sigma_n\} \in \Gamma_2$ and $\sum_{1\leq i<j\leq n} \sigma_i\sigma_j = 1$. Then $M = Df_2(\Sigma) = diag\{\eta_1, ..., \eta_n\} \in \mathcal{M}_2$ with

$$\eta_j = \frac{1}{2}(\sum_{i=1}^{n} \sigma_i - \sigma_j).$$

Now given a positive $t > 0$, we are trying to find a matrix

$$\tilde{\Sigma} = diag\{\tilde{\sigma_1}, \tilde{\sigma_2}, ..., \tilde{\sigma_n}\} \in \Gamma_2,$$

with $f_2(\tilde{\Sigma}) = 1$, such that

$$\tilde{M} = Df_2(\tilde{\Sigma}) = diag\{\tilde{\eta_1}, \tilde{\eta_2}, ..., \tilde{\eta_n}\}$$

with

$$\tilde{\eta_j} = \eta_j/t, j = 1, 2, ..., n-1,$$

and for a suitable function $h(t) > 0$ such that

$$\tilde{\eta_n} = h(t)\eta_n.$$

Here we require $h(t) > 0$ to make sure $\tilde{M} \in \mathcal{M}_2$.

The following $n-1$ equations

$$\sum_{i=1}^{n} \tilde{\sigma_i} - \tilde{\sigma_j} = 2\eta_j/t, j = 1, 2, ..., n-1$$

lead to

$$\tilde{\sigma_j} = \sigma_j/t - \frac{1}{n-2}(\tilde{\sigma_n} - \sigma_n/t), j = 1, 2, ..., n-1.$$

Then we can see the function $h(t)$

$$h(t) = \frac{\frac{1}{t}\sum_{j=1}^{n-1}\sigma_j - \frac{n-1}{n-2}\tilde{\sigma_n} + \frac{n-1}{t(n-2)}\sigma_n}{\sum_{j=1}^{n-1}\sigma_j}.$$

We try to calculate $\tilde{\sigma_n}$ by using the following equality

$$1 = \sum_{1\le i<j\le n}\tilde{\sigma_i}\tilde{\sigma_j}.$$

We can see

$$\begin{aligned}
1 &= \sum_{1\le i<j\le n-1}\tilde{\sigma_i}\tilde{\sigma_j} + \tilde{\sigma_n}\sum_{1\le j\le n-1}\tilde{\sigma_j} \\
&= \frac{1}{t^2}\sum_{1\le i<j\le n-1}\sigma_i\sigma_j + (\tilde{\sigma_n} - \sigma_n/t)^2\frac{n-1}{2(n-2)} \\
&- \frac{1}{t}(\tilde{\sigma_n} - \sigma_n/t)\sum_{j=1}^{n-1}\sigma_j + \tilde{\sigma_n}(\frac{1}{t}\sum_{j=1}^{n-1}\sigma_j - \frac{n-1}{n-2}(\tilde{\sigma_n} - \sigma_n/t)) \\
&= \frac{1}{t^2} + \frac{n-1}{2(n-2)}(\sigma_n^2/t^2 - \tilde{\sigma_n}^2).
\end{aligned}$$

Here we use the equality $\sum_{1\le i<j\le n}\sigma_i\sigma_j = 1$. Then

$$\tilde{\sigma_n} = \frac{1}{t}\sqrt{\sigma_n^2 + \frac{(1-t^2)2(n-2)}{n-1}}.$$

And we can calculate

$$\begin{aligned}
h(t) &= \frac{\frac{1}{t}\sum_{j=1}^{n-1}\sigma_j - \frac{n-1}{n-2}\tilde{\sigma_n} + \frac{n-1}{t(n-2)}\sigma_n}{\sum_{j=1}^{n-1}\sigma_j} \\
&= \frac{1}{t} - \frac{n-1}{t(n-2)}\frac{\sigma_n}{\sum_{j=1}^{n-1}\sigma_j}(\sqrt{1 + \frac{(1-t^2)2(n-2)}{(n-1)\sigma_n^2}} - 1).
\end{aligned} \tag{1.4.22}$$

We need $\tilde{M} \in \mathcal{M}_2$, and thus $\tilde{M}$ is a positive symmetric matrix. Therefore, we need $h(t) > 0$. By (1.4.22), $th(t) > 0$ is equivalent to

$$1 + \frac{2(n-2)}{(n-1)\sigma_n^2}(1-t^2) < (1 + \frac{n-2}{(n-1)\sigma_n}\sum_{j=1}^{n-1}\sigma_j)^2$$
$$= 1 + \frac{2(n-2)}{(n-1)\sigma_n}\sum_{j=1}^{n-1}\sigma_j + \frac{(n-2)^2}{(n-1)^2\sigma_n^2}(\sum_{j=1}^{n-1}\sigma_j^2 + 2\sum_{1\le i<j\le n-1}\sigma_i\sigma_j).$$

After simplification, it is equivalent to

$$1 - t^2 < \sigma_n\sum_{j=1}^{n-1}\sigma_j + \frac{n-2}{n-1}(\frac{1}{2}\sum_{j=1}^{n-1}\sigma_j^2 + \sum_{1\le i<j\le n-1}\sigma_i\sigma_j).$$

Since

$$1 = \sum_{1\le i<j\le n}\sigma_i\sigma_j = \sum_{1\le i<j\le n-1}\sigma_i\sigma_j + \sigma_n\sum_{j=1}^{n-1}\sigma_j,$$

we can get

$$0 < 2(n-1)t^2 + 2(\sigma_n\sum_{j=1}^{n-1}\sigma_j - 1) + (n-2)\sum_{j=1}^{n-1}\sigma_j^2$$
$$= 2(n-1)t^2 - 2\sum_{1\le i<j\le n}\sigma_i\sigma_j + (n-2)\sum_{j=1}^{n-1}\sigma_j^2$$
$$= 2(n-1)t^2 + \sum_{1\le i<j\le n}(\sigma_i - \sigma_j)^2,$$

and the right-hand side is strictly positive independently of $t > 0$. Therefore, we can prove $th(t) > 0$ for any $t > 0$.

Let $\tilde{\lambda}_j = \tilde{\eta}_j^{-1/2}$. Then

$$\tilde{\lambda}_j = \sqrt{t}\lambda_j, j = 1, 2, ..., n-1,$$

and

$$\tilde{\lambda}_n = \sqrt{h(t)}^{-1}\lambda_n.$$

Since $\tilde{M} \in \mathcal{M}_2$, then

$$
\begin{aligned}
0 < \frac{\eta_0}{1-s} &\leq \prod_{j=1}^{n} \tilde{\lambda}_j \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{(\tilde{\lambda}_1^2 y_1^2 + ... + \tilde{\lambda}_n^2 y_n^2)^{(n+2s)/2}} dy \\
&= \prod_{j=1}^{n} \tilde{\lambda}_j \int_{\mathbb{R}^n} \frac{u(\bar{y}, y_n) - u(\bar{y}, 0)}{(\tilde{\lambda}_1^2 y_1^2 + ... + \tilde{\lambda}_n^2 y_n^2)^{(n+2s)/2}} dy \\
&+ \prod_{j=1}^{n} \tilde{\lambda}_j \int_{\mathbb{R}^n} \frac{u(\bar{y}, 0) - u(0)}{(\tilde{\lambda}_1^2 y_1^2 + ... + \tilde{\lambda}_n^2 y_n^2)^{(n+2s)/2}} dy \\
&= P_1 + P_2.
\end{aligned} \tag{1.4.23}
$$

First we can calculate P_1. Let

$$
z_j = \frac{y_j}{|y_n|} \sqrt{th(t)} \lambda_j / \lambda_n, j = 1, 2, ..., n-1,
$$

and

$$
z_n = y_n,
$$

then

$$
\begin{aligned}
P_1 &\leq \frac{1}{2} \prod_{j=1}^{n} \lambda_j t^{(n-1)/2} \sqrt{h(t)}^{-1} \int_{\mathbb{R}^n} \frac{\min\{2L|z_n|, SC|z_n|^2\}}{(\lambda_n / \sqrt{h(t)})^{n+2s} (z_1^2 + ... + z_{n-1}^2 + 1)^{(n+2s)/2} |y_n|^{n+2s}} dz_n \\
&\cdot |y_n|^{n-1} (\frac{\lambda_n}{\sqrt{h(t)}})^{n-1} \sqrt{t}^{-(n-1)} \prod_{j=1}^{n-1} \lambda_j d\bar{z} \\
&= \lambda_n^{-2s} h(t)^s \frac{1}{2} \int_{\mathbb{R}} \frac{\min\{2L|z_n|, SC|z_n|^2\}}{|z_n|^{1+2s}} dz_n \int_{\mathbb{R}^{n-1}} \frac{1}{(1 + |\bar{z}|^2)^{\frac{n+2s}{2}}} d\bar{z} \\
&\leq C_1 C_2 h(t)^s \lambda_n^{-2s}.
\end{aligned}
$$

Here $C_1 = C_1(s, L, SC)$ and $C_2 = C_2(n, s)$ are given by (1.4.9) and (1.4.10) respectively.

Then we calculate P_2. Let

$$z_j = y_j, j = 1, 2, ..., n-1,$$

and

$$z_n = \frac{\lambda_n y_n}{\sqrt{th(t)}(\lambda_1^2 y_1^2 + ...\lambda_{n-1}^2 y_{n-1}^2)^{1/2}}.$$

Then

$$P_2 = \prod_{j=1}^{n} \lambda_j t^{(n-1)/2} \sqrt{h(t)}^{-1} \int_{\mathbb{R}^n} \frac{u(\bar{z}, 0) - u(0)}{t^{(n+2s)/2}(\lambda_1^2 z_1^2 + ... + \lambda_{n-1}^2 z_{n-1}^2)^{(n+2s)/2}(1 + z_n^2)^{(n+2s)/2}} d\bar{z}$$

$$\cdot \sqrt{th(t)}(\lambda_1^2 z_1^2 + ...\lambda_{n-1}^2 z_{n-1}^2)^{1/2} \frac{1}{\lambda_n} dz_n$$

$$= t^{-s} \lambda_n^{-1} \prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{(\lambda_1^2 z_1^2 + ... + \lambda_{n-1}^2 z_{n-1}^2)^{(n+2s-1)/2}} d\bar{z} \int_{\mathbb{R}} \frac{1}{(1 + z_n^2)^{(n+2s)/2}} dz_n$$

$$= C_3 t^{-s} \lambda_n^{-1} \prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{(\lambda_1^2 z_1^2 + ... + \lambda_{n-1}^2 z_{n-1}^2)^{(n+2s-1)/2}} d\bar{z},$$

with $C_3 = C_3(n, s)$ defined in (1.4.11).

Let

$$J = \prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z}, 0) - u(0)}{(\lambda_1^2 z_1^2 + ... + \lambda_{n-1}^2 z_{n-1}^2)^{(n+2s-1)/2}} d\bar{z},$$

then we get

$$C_1 C_2 h(t)^s \lambda_n^{-2s} + C_3 t^{-s} \lambda_n^{-1} J \geq \eta_0/(1-s) > 0. \tag{1.4.24}$$

Thus,

$$J \geq -\frac{C_1 C_2 h(t)^s \lambda_n^{-2s}}{C_3 t^{-s} \lambda_n^{-1}} = -\frac{C_1 C_2}{C_3}(th(t))^s \lambda_n^{1-2s}.$$

We have proved $th(t) > 0$ independently of $t > 0$, and thus for every $0 < t < 1$, equation (1.4.22) shows

$$0 < th(t) = 1 - \frac{n-1}{(n-2)} \frac{\sigma_n}{\sum_{j=1}^{n-1} \sigma_j} (\sqrt{1 + \frac{(1-t^2)2(n-2)}{(n-1)\sigma_n^2}} - 1) \leq 1.$$

Here we use $\sigma_n > 0$ and $\sum_{j=1}^{n-1} \sigma_j = 2\eta_n > 0$. Therefore, when $\lambda_n = \epsilon^{-1/2}$,

$$J \geq -\frac{C_1 C_2}{C_3}(th(t))^s \lambda_n^{1-2s} \geq -C\epsilon^{s-1/2},$$

and $\epsilon^{s-1/2} \to 0$ as $\epsilon \to 0$.

Exchange the positions of σ_{n-1} and σ_n in the matrix Σ, and this leads to the exchange of η_{n-1} and η_n. In the exact same way of constructing matrix $\tilde{M}$ and calculating the integral, we obtain an inequality

$$C_1 C_2 h(t)^s \lambda_{n-1}^{-2s} + C_3 t^{-s} \lambda_{n-1}^{-1} K \geq \eta_0/(1-s) > 0, \tag{1.4.25}$$

with

$$K = \prod_{j=1}^{n} \lambda_j \int_{\mathbb{R}^{n-1}} \frac{u(\bar{z},0) - u(0)}{(\lambda_1^2 z_1^2 + ... \lambda_{n-2} z_{n-2}^2 + \lambda_n^2 z_{n-1}^2)^{(n+2s-1)/2}} d\bar{z},$$

and

$$th(t) = 1 - \frac{n-1}{(n-2)} \frac{\sigma_{n-1}}{\sum_{j \neq n-1, 1 \leq j \leq n} \sigma_j} (\sqrt{1 + \frac{(1-t^2)2(n-2)}{(n-1)\sigma_{n-1}^2}} - 1).$$

The proof of $th(t) > 0$ independently of $t > 0$ remains the same since we only use $\sum_{1 \leq i < j \leq n} \sigma_i \sigma_j = 1$ in it. Then

$$K \geq -\frac{C_1 C_2}{C_3}(th(t))^s \lambda_{n-1}^{1-2s}.$$

We can see $\sigma_{n-1} > 0$ since $\sigma_1 \leq ... \leq \sigma_{n-1}$ and

$$0 < \eta_n = \frac{1}{2} \sum_{j=1}^{n-1} \sigma_j,$$

and we can see

$$\sum_{j\neq n-1,1\leq j\leq n}\sigma_j = 2\eta_{n-1} > 0.$$

Therefore, when $0 < t < 1$,

$$0 < th(t) = 1 - \frac{n-1}{(n-2)}\frac{\sigma_{n-1}}{\sum_{j\neq n-1,1\leq j\leq n}\sigma_j}\left(\sqrt{1+\frac{(1-t^2)2(n-2)}{(n-1)\sigma_{n-1}^2}}-1\right) \leq 1,$$

and by (1.4.25),

$$\begin{aligned}
&\prod_{j=1}^{n}\lambda_j\int_{\mathbb{R}^{n-1}}\frac{u(\bar{z},0)-u(0)}{(\lambda_1^2z_1^2+...\lambda_{n-2}z_{n-2}^2+\lambda_n^2z_{n-1}^2)^{(n+2s-1)/2}}d\bar{z}\\
&\geq -\frac{C_1C_2}{C_3}(th(t))^s\lambda_{n-1}^{1-2s}\\
&\geq -\frac{C_1C_2}{C_3}(\eta_{n-1})^{s-1/2}\\
&\geq -\frac{C_1C_2}{C_3}(\eta_1)^{s-1/2},
\end{aligned}$$

since $\eta_1 \geq \eta_2 \geq ... \geq \eta_{n-1} \geq \eta_n = \epsilon$. Then,

$$\begin{aligned}
I_2 &= \prod_{j=1}^{n}\lambda_j\int_{|\theta|\geq\theta_0}\int_0^\infty\int_{x\in\partial B_1^{n-1}(0),x\perp\tilde{e}(\theta)}\\
&\quad\frac{u(r(x_1\tilde{e}_1+...+x_{n-1}\tilde{e}_{n-1}))-u(0)}{r^{1+2s}(\lambda_1^2x_1^2+...+(\lambda_{n-1}^2\cos^2\theta+\lambda_n^2\sin^2\theta)x_{n-1}^2)^{\frac{n+2s}{2}}}dxdrd\theta\\
&= \int_{-\pi/2+\theta_0}^{\pi/2-\theta_0}\int_{\bar{z}\in\mathbb{R}^{n-1}}\frac{u(z_1\tilde{e_1}+...+z_{n-1}\tilde{e}_{n-1})-u(0)}{(\lambda_1^2z_1^2+...+(\lambda_{n-1}^2\cos^2\theta+\lambda_n^2\sin^2\theta)z_{n-1}^2)^{(n+2s-1)/2}}d\bar{z}d\theta\\
&\geq -(\pi-2\theta_0)\frac{C_1C_2}{C_3}(\eta_1)^{s-1/2}\\
&\geq -\pi\frac{C_1C_2}{C_3}(\eta_1)^{s-1/2}\\
&= -C_8\eta_1^{s-1/2}.
\end{aligned}$$

This completes the proof of Proposition 1.4.3.

□

Chapter 2

A non-local one-phase free boundary problem from obstacle to cavitation

2.1 Introduction

In this paper, we discuss the regularity properties of a free boundary problem of the following energy

$$J(u) = J_\gamma(u) = \frac{1}{2}\int_{(B_1^{n+1})^+} y^{1-2s}|\nabla u(x,y)|^2 dxdy + \int_{B_1^n\times\{y=0\}} u^\gamma dx, \quad (2.1.1)$$

with $0 < s, \gamma < 1$, subject to $u \geq 0$. The first part of the energy is related to the extension of the fractional Laplacian operator, and the second one is considered as a penalty for the function u being greater than 0. The set $\{u = 0\}$ only lies on $\{y = 0\}$, and is non-trivial if u is small enough on $\partial B_1^{n+1} \cap \{y > 0\}$. The boundary of the set $\{u > 0\}$ in the topology of $\mathbb{R}^n$ is called the free boundary. There is one important number $\beta = \frac{2s}{2-\gamma}$, which is the critical exponent in the scaling of the energy.

This problem is a non-local analogue of the problem introduced in [2] by Alt and Philips, in which a free boundary problem of the energy functional $\int_{B_1^n(0)} |\nabla u|^2 + |max(u,0)|^\gamma$ is discussed. We are now considering the case for the fractional Laplacian operator instead of Laplacian, and this is an intersection of one-phase free boundary problems and non-local integrodifferential operators. Heuristically, two limiting classical problems, one as $\gamma \to 0$ is the Bernoulli type one-phase free boundary problem from the minimization of $J_0(u) = \frac{1}{2}\int |\nabla u|^2 + \chi_{\{u>0\}}$, discussed by Alt and Caffarelli in [1]; and the other one as $\gamma \to 1$ is the obstacle problem from the minimimzation of $J_1(u) = \frac{1}{2}\int |\nabla u|^2 + max(u, 0)$, discussed by Caffarelli in [3]. Analogues of both problems in the fractional cases are also dicussed in [11] [13] [17] [16] for the Bernoulli type problems, and in [12] [6] for the thin obstacle problems. These are the inspirations for our minimization problem, which is an intermediate case of the fractional one-phase cavitation problem and thin obstacle problem.

There are some previous results on the properties of the minimizers of the energy $J_\gamma(u)$. In [20] by Yang, optimal regularity is proved, that the minimizer is C^β continuous if $\beta < 1$ and is C^α continuous for any $\alpha < 1$, if $\beta \geq 1$. The minimizer along the set $\{y = 0\}$ is C^β continuous if $\beta < 1$ and is $C^{1,\beta-1}$ continuous if $\beta \geq 1$. Non-degeneracy of the minimizer is also proved, that $\sup_{x \in B_r^n(x_0)} u(x, 0) \geq C(n, s, \gamma)r^\beta$ if $(x_0, 0)$ is a free boundary point.

This paper is divided into two parts. In the first part, we use Weiss type monotonicity formula introduced in [19] to prove the blow-up profiles are homogeneous of degree $\beta = \frac{2s}{2-\gamma}$. We also prove that the half-plane solution is unique up to rotation. The other part is to prove there exists a small constant

$\gamma_0 > 0$, such that for each $0 < \gamma < \gamma_0$, flatness condition of the free boundary implies $C^{1,\theta}$ regualrity, applying the method introduced in [13] by De Silva, Savin and Sire.

2.2 Preliminaries

Throughout this paper, we have the following notations. A point in the upper half space is $X = (x, y) \in (\mathbb{R}^{n+1})^+ = \mathbb{R}^n \times \mathbb{R}^+$; the upper half ball of radius R centered at 0 is $(B_R^{n+1})^+ = \{(x, y) \in (\mathbb{R}^{n+1})^+, |(x, y)| < R, y > 0\}$, its boundary in $\{y > 0\}$ is $(\partial B_R^{n+1})^+ = \{(x, y) \in (\mathbb{R}^{n+1})^+, |(x, y)| = R, y > 0\}$, and its boundary on $\{y = 0\}$ is $B_R^n = \{(x, y) \in (\mathbb{R}^{n+1})^+, |x| < R, y = 0\}$. Sometimes, we use B_1^+ instead of $(B_1^{n+1})^+$ for simplification.

We define $\alpha = 1 - 2s$ with $s \in (0, 1)$ the order of fractional Laplacian, and $\beta = \frac{2s}{2-\gamma}$ is the critical scaling exponent with $0 < \gamma < 1$.

Define the energy $J(u) = J_\gamma(u)$ by

$$J_\gamma(u) = \frac{1}{2} \int_{(B_1^{n+1})^+} y^{1-2s} |\nabla u(x, y)|^2 dx dy + \int_{B_1^n \times \{y=0\}} u^\gamma dx.$$

The set $\{u = 0\}$ which necessarily lies on $\{y = 0\}$ is called the contact set of u. Let the free boundary $F(u)$ denote the interface between the set $\{u > 0\} \cap \{y = 0\}$ and the contact set. We call $(x_0, 0)$ a free boundary point if $(x_0, 0) \in F(u)$.

2.2.1 Fractional Laplacian and Caffarelli-Silvestre extension

The fractional Laplacian is a non-local integral operator defined as

$$(-\Delta)^s u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy,$$

$$C_{n,s} = \frac{4^s \Gamma(n/2+s)}{\pi^{n/2}|\Gamma(-s)|},$$

with a corresponding non-local energy

$$E(u) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dydx$$

which is hard to handle. So an extension of the function to one extra dimension is introduced by Caffarelli and Silvestre in [8], transforming a non-local equation on $\mathbb{R}^n$ to an elliptic equation on the upper half space $\mathbb{R}^n \times \mathbb{R}^+$ with a Neumann boundary condition. Consider a fractional Laplacian equation $(-\Delta)^s u(x) = f(x)$ in $\mathbb{R}^n$, and $u \in H^s(\mathbb{R}^n)$. Define the extension $U(x, y)$ in $\mathbb{R}^n \times \mathbb{R}^+$ by a Poisson kernel in Section 2.4 in [8], such that $U(x, 0) = u(x)$ and the extension $U(x, y)$ satisfies the following equation with Neumann boundary condition,

$$div(y^{1-2s} \nabla U(x, y)) = 0 \quad \text{in} \quad \mathbb{R}^n \times \mathbb{R}^+ \tag{2.2.1}$$

and

$$\lim_{y \to 0^+} y^{1-2s} \partial_y U(x, y) = -C_{n,s}(-\Delta)^s u(x) \quad \text{in} \quad \mathbb{R}^n. \tag{2.2.2}$$

There is a natural energy

$$E(U) = \int_{\mathbb{R}^n \times \mathbb{R}^+} y^{1-2s} |\nabla U(x, y)|^2 dxdy$$

corresponding to the Euler-Lagrange equation (2.2.1).

From the Euler-Lagrange equation of the energy

$$J_\gamma(u) = \frac{1}{2}\int_{(B_1^{n+1})^+} y^{1-2s}|\nabla u(x,y)|^2 dxdy + \int_{B_1^n\times\{y=0\}} u^\gamma dx,$$

the minimizer u satisfies a second order PDE,

$$div(y^\alpha \nabla u) = 0$$

in the upper half ball $(B_1^{n+1})^+$ in a distributional sense, and

$$\lim_{y\to 0^+} y^\alpha \partial_y u(x,y) = \gamma u^{\gamma-1}(x,0)$$

on $\{u>0\}\cap\{y=0\}$. In the paper we let $\alpha = 1-2s$.

2.2.2 Scaling of the problem

Let $(0,0)$ be a free boundary point, and define the scaling $u_\lambda(X) = \lambda^{-\beta}u(\lambda X)$, $X=(x,y)\in(B_{\lambda^{-1}}^{n+1})^+$, then by the change of variables,

$$\begin{aligned} J((B_{\lambda^{-1}}^{n+1})^+, u_\lambda) &= \frac{1}{2}\int_{(B_{\lambda^{-1}}^{n+1})^+} y^\alpha \lambda^{-2\beta}|\nabla u(\lambda x, \lambda y)|^2 dxdy \\ &+ \int_{B_{\lambda^{-1}}^n\times\{y=0\}} \lambda^{-\beta\gamma} u^\gamma(\lambda x) dx \\ &= \frac{1}{2}\lambda^{-n+2-2\beta-\alpha}\int_{(B_1^{n+1})^+} y^\alpha |\nabla u(x,y)|^2 dxdy \\ &+ \lambda^{-n+1-\beta\gamma}\int_{B_1^n\times\{y=0\}} u^\gamma dx. \end{aligned}$$

We require two equal exponents of λ, and this leads to

$$\beta = \frac{2s}{2-\gamma} = \frac{1-\alpha}{2-\gamma},$$

and thus

$$J((B^{n+1}_{\lambda^{-1}})^+, u_\lambda) = \lambda^{-n+1-\beta\gamma} J((B^{n+1}_1)^+, u).$$

So if u is a minimizer for the energy in $(B^{n+1}_1)^+$, then u_λ is a minimizer in $(B^{n+1}_{\lambda^{-1}})^+$.

2.2.3 Function space

We are considering minimizers of energy

$$J_\gamma(u) = \frac{1}{2}\int_{(B^{n+1}_1)^+} y^\alpha |\nabla u(x,y)|^2 dxdy + \int_{B^n_1 \times \{y=0\}} u^\gamma dx$$

in the space $H^1(y^\alpha, B^+_1)$, which is a weighted H^1 space, with norm

$$\|u\|_{H^1(y^\alpha, B^+_1)} = (\int_{(B^{n+1}_1)^+} y^\alpha (|\nabla u|^2 + u^2) dxdy)^{1/2},$$

and seminorm

$$[u]_{H^1(y^\alpha, B^+_1)} = (\int_{(B^{n+1}_1)^+} y^\alpha |\nabla u|^2 dxdy)^{1/2}.$$

From the extension theorem of Caffarelli and Silverstre in [8], trace of any $H^1(y^\alpha, B^+_1)$ function lies in $H^s(B^n_1(0))$, and by Sobolev embedding, the trace also lies in $L^2(B^n_1(0))$.

2.2.4 Blow-up limits

We define the scaling of the minimizer near a free boundary point $(x_0, 0)$ by

$$u_R(x,y) = \frac{u(R(x-x_0)+x_0, Ry)}{R^\beta}.$$

There is a subsequence u_{R_k} converging to u_0,

$$u_0(x,y) = \lim_{R_k \to 0} u_{R_k}(x,y),$$

and the limit u_0 is the blow-up profile near point $(x_0, 0)$ on the free boundary.

Lemma 2.2.1. *Let $\{u_n\}$ be a sequence of minimizers of J, bounded in $H^1(y^\alpha, B_1^+)$. Then any (weakly) converging subsequence of $\{u_n\}$ converges to a minimizer of J in B_1^+.*

The proof of Lemma 2.2.1 is the same as the proof of Lemma 1.14 in the classical paper [2], with the optimal C^β continuity of the minimizers proved in [20] and their boundedness in $H^1(y^\alpha, B_1^+)$.

A global minimizer is a function $u \in H^1_{loc}(y^\alpha, (\mathbb{R}^{n+1})^+)$ which minimizes the energy J in every $B_R^{n+1} \cap \{y \geq 0\}$. As a consequence of Lemma 2.2.1 and the fact that all the blow-ups u_r are the rescaling of the same function, we can obtain the following corollary.

Corollary 2.2.2. *Let u be the minimizer of energy J in $(B_1^{n+1})+$, and let $(0,0)$ be a free boundary point. Then u_r is a local minimizer of J in $(B_{1/r}^{n+1})^+$, and any uniform limit of the family u_r is a global minimizer of J in $(\mathbb{R}^{n+1})^+$.*

2.3 Blow-ups are homogeneous of degree β

In this section, we use Weiss type monotonicity formula to prove the blow-up of the energy minimizer at every free boundary point is homogeneous of degree β.

If u is a minimizer of the energy $J(u)$, then it satisfies

$$div(y^\alpha \nabla u) = 0 \text{ in } (B_1^{n+1})^+, \tag{2.3.1}$$

and

$$\lim_{y\to 0} y^\alpha \partial_y u(x,y) = \gamma u^{\gamma-1}(x,0) \text{ on } B_1^n \cap \{u > 0\}. \tag{2.3.2}$$

Here we introduce a boundary adjusted energy and define

$$\begin{aligned} W(R,u) = {} & R^{-(n-1+\frac{2-\alpha\gamma}{2-\gamma})} \int_{(B_R^{n+1})^+} y^\alpha |\nabla u|^2 dxdy \\ & + 2R^{-(n-1+\frac{2-\alpha\gamma}{2-\gamma})} \int_{B_R^n} u^\gamma dx \\ & - \beta R^{-(n+\frac{2-\alpha\gamma}{2-\gamma})} \int_{(\partial B_R^{n+1})^+} y^\alpha u^2 d\sigma. \end{aligned} \tag{2.3.3}$$

This energy is invariant under scaling,

$$W(R\rho, u) = W(\rho, u_R), \tag{2.3.4}$$

with

$$u_R(x,y) = \frac{u(Rx,Ry)}{R^\beta}.$$

Theorem 2.3.1 (Weiss type monotonicity formula)**.** *If u is a minimizer of $J(u)$ and $(0,0)$ is a free boundary point, then the boundary adjusted energy $W(R,u)$ satisfies the monotonicity formula*

$$\frac{d}{dR} W(R,u) = 2R^{-(n-1+\frac{2-\alpha\gamma}{2-\gamma})} \int_{(\partial B_R^{n+1})^+} y^\alpha (u_\nu - \beta \frac{u}{R})^2 d\sigma, \tag{2.3.5}$$

with ν the outer normal. Moreover, when $\frac{d}{dR}W(R,u)=0$, it is equivalent that

$$0=<(x,y),\nabla u(x,y)>-\beta u(x,y)=\frac{d}{d\rho}|_{\rho=1}\frac{u(\rho x,\rho y)}{\rho^{\beta}}$$

a.e. on $(\partial B_1^{n+1})^+$, which means u is homogeneous of degree β.

Proof. If u is a minimizer of the energy $J(u)$, then it satisfies $div(y^{\alpha}\nabla u)=0$, $div(y^{\alpha}u\nabla u)=y^{\alpha}|\nabla u|^2$ in $(B_1^{n+1})^+$and $\lim_{y\to 0}y^{\alpha}\partial_y u(x,y)=\gamma u^{\gamma-1}(x,0)$ on $B_1^n\cap\{u>0\}$. Then the following equalities are obtained:

$$\int_{(B_R^{n+1})^+}y^{\alpha}|\nabla u|^2dxdy=\int_{(\partial B_R^{n+1})^+}y^{\alpha}uu_{\nu}d\sigma-\gamma\int_{B_R^n}u^{\gamma}dx; \tag{2.3.6}$$

$$\int_{B_R^n}<x,\nabla u^{\gamma}>dx=R\int_{\partial B_R^n}u^{\gamma}d\sigma-n\int_{B_R^n}u^{\gamma}dx; \tag{2.3.7}$$

$$\frac{d}{dR}(\int_{(\partial B_R^{n+1})^+}y^{\alpha}u^2d\sigma)=\frac{n+\alpha}{R}\int_{(\partial B_R^{n+1})^+}y^{\alpha}u^2d\sigma+2\int_{(\partial B_R^{n+1})^+}y^{\alpha}uu_{\nu}d\sigma; \tag{2.3.8}$$

$$\begin{aligned}&(n+\alpha-1)\int_{(B_R^{n+1})^+}y^{\alpha}|\nabla u|^2dxdy\\&=R\int_{(\partial B_R^{n+1})^+}y^{\alpha}(|\nabla u|^2-2u_{\nu}^2)d\sigma+2\int_{B_R^n}<x,\nabla u^{\gamma}>dx\\&=R\int_{(\partial B_R^{n+1})^+}y^{\alpha}(|\nabla u|^2-2u_{\nu}^2)d\sigma+2R\int_{\partial B_R^n}u^{\gamma}d\sigma-2n\int_{B_R^n}u^{\gamma}dx.\end{aligned} \tag{2.3.9}$$

Compute the derivative of $W(R,u)$ with respect to R and we can get:

$$\begin{aligned}
R^{n+\frac{2-\alpha\gamma}{2-\gamma}}\frac{d}{dR}W(R,u) = &-(n-1+\frac{2-\alpha\gamma}{2-\gamma})\int_{(B_R^{n+1})^+} y^\alpha|\nabla u|^2 dxdy \quad (I_1)\\
&+R\int_{(\partial B_R^{n+1})^+} y^\alpha|\nabla u|^2 d\sigma \quad (I_2)\\
&-2(n+\frac{\gamma-\alpha\gamma}{2-\gamma})\int_{B_R^n} u^\gamma dx \quad (I_3)\\
&+2R\int_{\partial B_R^n} u^\gamma d\sigma \quad (I_4)\\
&+\beta(n+\frac{2-\alpha\gamma}{2-\gamma})R^{-1}\int_{(\partial B_R^{n+1})^+} y^\alpha u^2 d\sigma \quad (I_5)\\
&-\beta\frac{n+\alpha}{R}\int_{(\partial B_R^{n+1})^+} y^\alpha u^2 d\sigma \quad (I_6)\\
&-2\beta\int_{(\partial B_R^{n+1})^+} y^\alpha u u_\nu d\sigma. \quad (I_7)
\end{aligned}$$

Apply (2.3.9) and (2.3.6), then

$$\begin{aligned}
(I_1)+(I_2) = &\, 2R\int_{(\partial B_R^{n+1})^+} y^\alpha u_\nu^2 d\sigma\\
&-2R\int_{\partial B_R^n} u^\gamma + 2n\int_{B_R^n} u^\gamma\\
&-\frac{2-2\alpha}{2-\gamma}(\int_{(\partial B_R^{n+1})^+} y^\alpha u u_\nu d\sigma - \gamma\int_{B_R^n} u^\gamma dx).
\end{aligned}$$

After adding (I_3) and (I_4) we obtain:

$$\begin{aligned}
(I_1)+(I_2)+(I_3)+(I_4) = &\, 2R\int_{(\partial B_R^{n+1})^+} y^\alpha u_\nu^2 d\sigma\\
&-\frac{2-2\alpha}{2-\gamma}\int_{(\partial B_R^{n+1})^+} y^\alpha u^2 d\sigma,
\end{aligned}$$

Adding the last three terms (I_5), (I_6) and (I_7), we can compute that

$$\frac{d}{dR}W(R,u) = 2R^{-(n-1+\frac{2-\alpha\gamma}{2-\gamma})}\int_{(\partial B_R^{n+1})^+} y^\alpha(u_\nu-\beta\frac{u}{R})^2 d\sigma.$$

$\square$

Let $0 \in \partial\{u > 0\} \cap \{y = 0\}$, and consider the function $u_r(X) = r^{-\beta}u(rX)$. As $r_k \to 0$, u_{r_k} converges to u_0 weakly in $H^1(y^\alpha, (\mathbb{R}^{n+1})^+)$. Pass to a subsequence (still denoted by u_{r_k}), $u_{r_k} \to u_0$ in $L^2_{loc}(y^\alpha, (\mathbb{R}^{n+1})^+)$, and in $L^2_{loc}(\mathbb{R}^n \times \{y = 0\})$. Thus, $W(r_k, u) = W(1, u_{r_k})$ is a bounded non-decreasing function of r_k by Theorem 2.3.1, if u is a minimizer. Then we can prove the following corollary.

Corollary 2.3.2 (Blow-ups are homogeneous of degree β)**.** *If u is a minimizer of $J(u)$, then the blow-up limit u_0 at every free boundary point is homogeneous of degree β.*

Proof. Since $W(\rho r, u) = W(\rho, u_r)$ by the scaling property of W, and $W(Rr_k, u)$ is a bounded non-decreasing function of r_k by Theorem 2.3.1, for any $R > 0$,

$$W(R, u_0) = \lim_{k\to\infty} W(R, u_{r_k}) = \lim_{k\to\infty} W(Rr_k, u) = W(0^+, u)$$

is a constant. Thus,

$$\frac{d}{dR}W(R, u_0) = 0,$$

and this implies that u_0 is homogeneous of degree β. $\square$

2.4 Uniqueness of the half-plane solution

In this section, we apply the method introduced in [11] to prove the following theorem.

Theorem 2.4.1. *If u is the minimizer of J_γ in $(\mathbb{R}^{n+1})^+$, and $u(x,0) = A(x_n)_+^\beta$, then*

$$A = A(s,\gamma) = \left(\frac{2(\beta - s)}{-\beta A_1}\right)^{1/(2-\gamma)} \tag{2.4.1}$$

is determined by s and γ, where

$$A_1 = -\frac{C_{1,s}}{2}\int_{-\infty}^{\infty} \frac{(1+y)_+^\beta + (1-y)_+^\beta - 2}{|y|^{1+2s}} dy < 0,$$

with constant

$$C_{1,s} = \frac{4^s \Gamma(1/2+s)}{\pi^{1/2}|\Gamma(-s)|}.$$

Proof. First we prove the theorem when $n = 1$. Let

$$J(u) = \frac{1}{2}\int_{(B_1^2)^+} y^\alpha |\nabla u|^2 dxdy + \int_{-1}^{1} u^\gamma dx,$$

and consider $U_0(x,y)$ as the extension of $u_0(x) = (x)_+^\beta$. Define

$$u_\epsilon(x) = \frac{(x+\epsilon)_+^\beta}{(1+\epsilon)^\beta},$$

and

$$\tilde{u}_\epsilon = \begin{cases} u_\epsilon(x) & |x| \le 1 \\ u_0(x) & |x| > 1. \end{cases}$$

Let function $U_\epsilon(x,y)$ satisfies the following equations:

$$\begin{cases} div(y^\alpha \nabla U_\epsilon(x,y)) = 0 \quad in \quad (B_1^2)^+ \\ U_\epsilon(x,0) = u_\epsilon(x) \qquad |x| \le 1 \\ U_\epsilon(x,y) = U_0(x,y) \quad on \quad (\partial B_1^2)^+. \end{cases}$$

If AU_0 is a local minimizer of $J(u)$, then $J(AU_0) \le J(AU_\epsilon)$ for every ϵ, positive and negative, then

$$\frac{1}{2}A^2\int_{(B_1^2)^+} y^\alpha|\nabla U_0|^2 dxdy + A^\gamma \int_{-1}^{1} u_0^\gamma dx \le \frac{1}{2}A^2\int_{(B_1^2)^+} y^\alpha|\nabla U_\epsilon|^2 dxdy + A^\gamma\int_{-1}^{1} u_\epsilon^\gamma dx.$$

We can see

$$\int_{-1}^{1} u_\epsilon^\gamma dx - \int_{-1}^{1} u_0^\gamma dx = \frac{1}{(1+\epsilon)^{\beta\gamma}}\frac{1}{1+\beta\gamma}(1+\epsilon)^{\beta\gamma+1} - \frac{1}{1+\beta\gamma}$$
$$= \frac{\epsilon}{1+\beta\gamma},$$

and

$$(-1)[\int_{(B_1^2)^+} y^\alpha|\nabla U_0|^2 - \int_{(B_1^2)^+} y^\alpha|\nabla U_\epsilon|^2]$$
$$= \int_{(B_1^2)^+} y^\alpha|\nabla(U_0 - U_\epsilon)|^2 + 2\int_{(B_1^2)^+} y^\alpha \nabla U_0 \nabla(U_\epsilon - U_0)$$
$$= I_2 + 2I_1.$$

First let us compute I_1:

$$I_1 = \int_{(B_1^2)^+} y^\alpha \nabla U_0 \nabla(U_\epsilon - U_0)$$
$$= \int_{(B_1^2)^+} div(y^\alpha \nabla U_0(U_\epsilon - U_0)) - \int_{(B_1^2)^+} ((U_\epsilon - U_0) div(y^\alpha \nabla U_0)$$
$$= \int_{(\partial B_1^2)^+} y^\alpha (U_0)_\nu (U_\epsilon - U_0) - \int_{-1}^{1} (\lim_{y\to 0^+} y^\alpha \partial_y U_0)(U_\epsilon - U_0)$$
$$= \int_{-1}^{1} (-\Delta)^s u_0(x)(u_\epsilon - u_0).$$

By the homogeneity property of u_0, we can compute that when $x > 0$,

$$(-\Delta)^s u_0(x) = -\frac{C_{1,s}}{2}\int_{-\infty}^{\infty} \frac{(x+y)_+^\beta + (x-y)_+^\beta - 2(x)_+^\beta}{|y|^{1+2s}} dy$$
$$= x^{\beta-2s}\frac{-C_{1,s}}{2}\int_{-\infty}^{\infty} \frac{(1+y)_+^\beta + (1-y)_+^\beta - 2}{|y|^{1+2s}} dy$$
$$= A_1 x^{\beta-2s},$$

and when $x < 0$,

$$(-\Delta)^s u_0(x) = -(-x)^{\beta-2s} C_{1,s} P.V. \int_{1}^{\infty} \frac{(y-1)^\beta}{|y|^{1+2s}} dy$$
$$= A_2(-x)^{\beta-2s}.$$

Notice that $A_1, A_2 < 0$, with

$$A_1 = -\frac{C_{1,s}}{2}\int_{-\infty}^{\infty} \frac{(1+y)_+^{\beta} + (1-y)_+^{\beta} - 2}{|y|^{1+2s}} dy,$$

and

$$A_2 = -C_{1,s} P.V. \int_1^{\infty} \frac{(y-1)^{\beta}}{|y|^{1+2s}} dy.$$

Then we can compute that

$$\begin{aligned} I_1 &= \int_{-1}^{1} (-\Delta)^s u_0(x)(u_\epsilon - u_0) \\ &= A_1 \int_0^1 x^{\beta-2s}\left(\frac{(x+\epsilon)_+^{\beta}}{(1+\epsilon)^{\beta}} - x^{\beta}\right) dx \\ &+ A_2 \int_{-1}^{0} (-x)^{\beta-2s} \frac{(x+\epsilon)_+^{\beta}}{(1+\epsilon)^{\beta}} dx \\ &= A_1 \beta\left(\frac{1}{2\beta-2s} - \frac{1}{2\beta-2s+1}\right)\epsilon + o(\epsilon). \end{aligned}$$

Then we compute I_2,

$$\begin{aligned} I_2 &= \int_{(B_1^2)^+} y^{\alpha} |\nabla(U_0 - U_\epsilon)|^2 \\ &= \int_{(B_1^2)^+} div(y^{\alpha}(U_\epsilon - U_0)\nabla(U_\epsilon - U_0)) - \int_{(B_1^2)^+} (U_\epsilon - U_0) div(y^{\alpha}\nabla(U_\epsilon - U_0)) \\ &= \int_{(\partial B_1^2)^+} y^{\alpha}(U_\epsilon - U_0)(U_\epsilon - U_0)_\nu - \int_{-1}^{1} (\lim_{y \to 0^+} y^{\alpha}\partial_y(U_\epsilon - U_0))(U_\epsilon - U_0) \\ &= \int_{-1}^{1} (-\Delta)^s(\tilde{u}_\epsilon - u_0)(u_\epsilon - u_0) \\ &= \int_{-1}^{1} (-\Delta)^s(\tilde{u}_\epsilon - u_\epsilon)(u_\epsilon - u_0) + \int_{-1}^{1} (-\Delta)^s(u_\epsilon - u_0)(u_\epsilon - u_0). \end{aligned}$$

Define

$$g_\epsilon(x) = \tilde{u}_\epsilon(x) - u_\epsilon(x) = \epsilon h(x) = \begin{cases} 0 & x \le 1 \\ \epsilon\beta(x^{\beta} - x^{\beta-1}) + o(\epsilon) & x > 1, \end{cases}$$

and

$$(-\Delta)^s(\tilde{u}_\epsilon - u_\epsilon)(x) = \epsilon C_{1,s} P.V. \int_{-\infty}^{\infty} \frac{h(x+y) - h(x)}{|y|^{1+2s}} dy,$$

and

$$\int_{-1}^{1} (-\Delta)^s(\tilde{u}_\epsilon - u_\epsilon)(u_\epsilon - u_0) \leq 2max|u_\epsilon - u_0| O(\epsilon) = o(\epsilon).$$

Thus,

$$I_2 = o(\epsilon) + \int_{-1}^{1} (-\Delta)^s u_\epsilon (u_\epsilon - u_0) - \int_{-1}^{1} (-\Delta)^s u_0 (u_\epsilon - u_0) = o(\epsilon) + I_3 - I_1,$$

where

$$I_3 = \int_{-1}^{1} (-\Delta)^s u_\epsilon (u_\epsilon - u_0).$$

Since $u_\epsilon(x) = \frac{(x+\epsilon)_+^\beta}{(1+\epsilon)^\beta}$,

$$(-\Delta)^s u_\epsilon(x) = \begin{cases} \frac{1}{(1+\epsilon)^\beta} A_1 (x+\epsilon)^{\beta-2s} & x+\epsilon > 0 \\ \frac{1}{(1+\epsilon)^\beta} A_2 (-x-\epsilon)^{\beta-2s} & x+\epsilon < 0, \end{cases}$$

and we can compute I_3 that

$$\begin{aligned} I_3 &= \int_{-1}^{1} (-\Delta)^s u_\epsilon (u_\epsilon - u_0) \\ &= \int_0^1 \frac{1}{(1+\epsilon)^\beta} A_1 (x+\epsilon)^{\beta-2s} \left(\frac{(x+\epsilon)^\beta}{(1+\epsilon)^\beta} - x^\beta\right) dx \\ &+ \int_{-\epsilon}^{0} \frac{1}{(1+\epsilon)^\beta} A_1 (x+\epsilon)^{\beta-2s} \frac{(x+\epsilon)^\beta}{(1+\epsilon)^\beta} dx \\ &= \epsilon A_1 \left(\frac{\beta-2s+1}{2\beta-2s+1} - \frac{\beta-2s}{2\beta-2s}\right) + o(\epsilon) \\ &= \epsilon A_1 \beta \left(\frac{1}{2\beta-2s} - \frac{1}{2\beta-2s+1}\right) + o(\epsilon). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{2} A^2 (I_2 + 2I_1) &= \frac{-1}{2} A^2 \left(\int y^\alpha |\nabla U_0|^2 - \int y^\alpha |\nabla U_\epsilon|^2\right) \\ &= -\epsilon A^2 A_1 \beta \left(\frac{1}{2\beta-2s} - \frac{1}{2\beta-2s+1}\right) + o(\epsilon), \end{aligned}$$

and

$$A^\gamma \int_{-1}^{1} u_\epsilon^\gamma dx - A^\gamma \int_{-1}^{1} u_0^\gamma dx = A^\gamma \frac{\epsilon}{1+\beta\gamma},$$

and since AU_0 is a local minimizer of energy $J(u)$, it is required that for all small $\epsilon > 0$ and $\epsilon < 0$,

$$-\epsilon A^2 A_1 \beta (\frac{1}{2\beta - 2s} - \frac{1}{2\beta - 2s + 1}) + o(\epsilon) \leq A^\gamma \frac{\epsilon}{1+\beta\gamma},$$

which indicates

$$A = (\frac{2(\beta - s)}{-\beta A_1})^{1/(2-\gamma)},$$

and A is determined by s and γ, where

$$A_1 = -\frac{C_{1,s}}{2} \int_{-\infty}^{\infty} \frac{(1+y)_+^\beta + (1-y)_+^\beta - 2}{|y|^{1+2s}} dy < 0.$$

Moreover, as $\gamma \to 0$, which is the case of fractional one-phase Bernoulli-type problem, the constant $A_1 = O(\beta - s)$ and this ensures the unique half plane minimizer is not 0 or ∞.

Applying the same proof in Theorem 1.4 in [11], we prove the theorem for general n. $\square$

2.5 Positive density when γ is small enough

When $\gamma \to 1$, in the thin obstacle problem [12], near a free boundary point $(x_0, 0)$, the set $\{u = 0\} \cap B_1^n(x_0)$ does not always have positive density. In this section, we prove there exists a positive number $\gamma_0 > 0$, and for each

$0 < \gamma < \gamma_0$, the minimizer of energy $J_\gamma(u)$ has positive density of zero set near every free boundary point.

Theorem 2.5.1. *There exists $\gamma_0 = \gamma_0(n, s) > 0$ and $\delta > 0$ such that for each $0 < \gamma < \gamma_0$, if u_γ is a minimizer of $J_\gamma(u)$, then*

$$L^n(\{u_\gamma = 0\} \cap B_1^n) \geq \delta > 0.$$

We prove the theorem by the method of compactness. Before the proof, a lemma of non-degeneracy is required.

Lemma 2.5.2. *Assume u_γ is a minimizer of the energy $J_\gamma(u)$ and $(0, 0)$ is a free boundary point. There exists a positive constant $C_0 > 0$ independent of γ, such that for each $x \in B_{1/2}^n \cap \{u > 0\}$,*

$$u_\gamma(x, 0) \geq C_0(d(x, \partial\{u_\gamma > 0\}))^\beta.$$

Proof. Up to rescaling, it is enough to show, if $(x_0, 0)$ is at distance 1 from the free boundary and $u_\gamma(x_0, 0) > 0$, then $\epsilon = u_\gamma(x_0, 0)$ cannot be too small, and ϵ does not converge to 0 as $\gamma \to 0$.

By the Harnack inequality in the upper half space (since y^α belongs to the class of A_2 functions defined by Muchenhoupt in [15]) and the variant boundary Harnack inequality proved in Theorem 4.1 in [20], there exists $c', C' > 0$ independent of γ, such that

$$0 < c'\epsilon \leq u_\gamma(x, y) \leq C'\epsilon$$

in $(B^{n+1}_{1/2}(x_0,0))^+$. Take test functions $\phi \in C^\infty_C((B^{n+1}_{1/2}(x_0,0))^+)$. We can compute

$$\begin{aligned}
&\int_{(B^{n+1}_{1/2}(x_0,0))^+} div(y^\alpha \phi \nabla u_\gamma) \\
&= \int_{(\partial B^{n+1}_{1/2}(x_0,0))^+} y^\alpha \phi (u_\gamma)_\nu - \int_{\{u_\gamma>0\}\cap B^n_{1/2}(x_0)} \phi(\lim_{y\to 0^+} y^\alpha \partial_y u_\gamma) \\
&= -\int_{\{u_\gamma>0\}\cap B^n_{1/2}(x_0)} \phi\gamma u_\gamma^{\gamma-1}.
\end{aligned}$$

We can see

$$\begin{aligned}
&\int_{(B^{n+1}_{1/2}(x_0,0))^+} y^\alpha \nabla u_\gamma \nabla \phi \\
&= \int_{(B^{n+1}_{1/2}(x_0,0))^+} div(y^\alpha u_\gamma \nabla \phi) - \int_{(B^{n+1}_{1/2}(x_0,0))^+} u_\gamma div(y^\alpha \nabla \phi) \\
&= \int_{(B^{n+1}_{1/2}(x_0,0))^+} div(y^\alpha \phi \nabla u_\gamma) - \int_{(B^{n+1}_{1/2}(x_0,0))^+} \phi div(y^\alpha \nabla u_\gamma).
\end{aligned}$$

Using $div(y^\alpha \nabla u_\gamma) = 0$, w obtain the following equality:

$$\begin{aligned}
&-\int_{\{u_\gamma>0\}\cap B^n_{1/2}(x_0)} \phi\gamma u_\gamma^{\gamma-1} \\
&= \int_{(B^{n+1}_{1/2}(x_0,0))^+} div(y^\alpha u_\gamma \nabla \phi) - \int_{(B^{n+1}_{1/2}(x_0,0))^+} u_\gamma div(y^\alpha \nabla \phi) \\
&= \int_{(\partial B^{n+1}_{1/2}(x_0,0))^+} y^\alpha u_\gamma (\phi)_\nu - \int_{\{u_\gamma>0\}\cap B^n_{1/2}(x_0)} u_\gamma(\lim_{y\to 0^+} y^\alpha \partial_y \phi) \\
&- \int_{(B^{n+1}_{1/2}(x_0,0))^+} u_\gamma div(y^\alpha \nabla \phi).
\end{aligned}$$

Then

$$
\begin{aligned}
|\int_{\{u_\gamma>0\}\cap B^n_{1/2}(x_0)} \gamma\phi(C'\epsilon)^{\gamma-1}| &\le |\int_{\{u_\gamma>0\}\cap B^n_{1/2}(x_0)} \gamma\phi u_\gamma^{\gamma-1}| \\
&\le |\int_{\{u>0\}\cap B^n_{1/2}(x_0)} u_\gamma(\lim_{y\to 0^+} y^\alpha \partial_y \phi)| \\
&+ |\int_{(B^{n+1}_{1/2}(x_0,0))^+} u_\gamma div(y^\alpha \nabla\phi)| \\
&+ |\int_{(\partial B^{n+1}_{1/2}(x_0,0))^+} y^\alpha u_\gamma (\phi)_\nu|.
\end{aligned} \tag{2.5.1}
$$

Since $d(x, \partial\{u_\gamma > 0\}) \le C$, if $(x,y) \in (B^{n+1}_{1/2}(x_0,0))^+$, then

$$u(x,y) \le \tilde{C}$$

by C^β estimates of the minimizer. The test function $\phi \in C^\infty_C((B^{n+1}_1(x_0,0))^+)$ is smooth enough, so the integral of $\lim_{y\to 0^+} y^\alpha \partial_y\phi$, $y^\alpha(\phi)_\nu$ and $div(y^\alpha\nabla\phi)$ are all bounded, and therefore by (2.5.1), ϵ cannot be too small.

However, $\gamma\epsilon^{\gamma-1} < \infty$ cannot ensure $\epsilon \ge C_0 > 0$ as $\gamma \to 0$. To prove that $\epsilon \ge C_0$ independent of γ, we consider a smooth function $P(x,y) \ge 0$ defined on $(B^{n+1}_{1/2}(x_0,0))^+$, with $P(x,y) = 0$ in $(B^{n+1}_{1/4}(x_0,0))^+$ and $P(x,y) = 2C'$ in $(B^{n+1}_{7/16}(x_0,0))^+ \setminus (B^{n+1}_{3/8}(x_0,0))^+$. Define a function

$$v(x,y) = \min\{u(x,y), \epsilon P(x,y)\} \quad \text{on} \quad (B^{n+1}_{1/2}(x_0,0))^+.$$

Then $J(v) \ge J(u)$ since $u(x,y)$ is the energy minimizer. First we can see

$$\int_{(B^{n+1}_{1/2}(x_0,0))^+} y^\alpha |\nabla v|^2 dxdy - \int_{(B^{n+1}_{1/2}(x_0,0))^+} y^\alpha |\nabla u|^2 dxdy \le O(\epsilon)$$

from our definition of the function $v(x,y)$. (Same as in Section 3.4, proof of Theorem 1.2 in [11]). Then

$$\int_{B^n_{1/2}(x_0)} v^\gamma - u^\gamma \leq -\int_{B^n_{1/4}(x_0)} u^\gamma,$$

since $v = 0$ on $B^n_{1/4}(x_0)$ and $v \leq u$ on $B^n_{1/2}(x_0)$. Therefore,

$$J(v) - J(u) \leq O(\epsilon) - \int_{B^n_{1/4}(x_0)} u^\gamma. \tag{2.5.2}$$

However, $J(v) \geq J(u)$ since u is the energy minimizer. Therefore, if $\epsilon \to 0$ as $\gamma \to 0$, then (2.5.2) requires $\epsilon^\gamma \to 0$ as $\gamma \to 0$. If not, (2.5.2) leads to a contradiction of u being the energy minimizer. Therefore, now it is required that, if $\epsilon \to 0$ as $\gamma \to 0$, then

$$\lim_{\gamma \to 0} \epsilon^\gamma = 0$$

and

$$\lim_{\gamma \to 0} \gamma \epsilon^{\gamma - 1} < \infty$$

from (2.5.2) and (2.5.1).

The first limit shows $\epsilon = e^{-\frac{1}{\gamma o(\gamma)}}$, and then as $\gamma \to 0$.

$$\gamma \epsilon^{\gamma-1} = \gamma e^{\frac{1}{\gamma o(\gamma)} - \frac{1}{o(\gamma)}} \to \gamma e^{\frac{1}{\gamma o(\gamma)}} \to \infty$$

Thus ϵ does not converge to 0 as $\gamma \to 0$, and therefore, $\epsilon \geq C_0 > 0$ independent of γ. $\square$

With the non-degeneracy property of the minimizer, we can prove Theorem 2.5.1 by the method of compactness.

Proof. If not, then there exists $\gamma_k \to 0$ with $\{u^j_{\gamma_k}\}_{j=1}^{\infty}$ a sequence of minimizers of J_{γ_k}, and

$$\lim_{\gamma_k \to 0, j\to\infty} L^n(\{u^j_{\gamma_k} = 0\} \cap B^n_1) = 0. \tag{2.5.3}$$

Without loss of generality, we assume $(0,0)$ is a common free boundary point and take the blow-up limit at point 0. Let $u^j_0 = \lim_{\gamma_k \to 0} u^j_{\gamma_k}$. By the $\Gamma-$convergence of

$$J_\gamma(u) \to J_0(u) = \int_{(B^{n+1}_1)^+} y^\alpha |\nabla u|^2 + \int_{B^n_1} \chi_{\{u>0\}},$$

we know $\{u^j_0\}_{j=1}^{\infty}$ is a sequence of minimizers of $J_0(u)$. Then Lemma 2.5.2 and (2.5.3) imply

$$\lim_{j\to\infty} L^n(\{u^j_0 = 0\} \cap B^n_1) = 0$$

which leads to a contradiction, since in Theorem 1.3 in [11] the authors prove that in the fractional cavitation problem, near every free boundary point, the zero set has positive density. □

2.6 Flatness to regularity preliminaries and the main theorem

In the following sections we apply the method introduced in [13] by De Silva, Savin and Sire to prove the regularity of free boundary given flatenss conditon when $0 < \gamma < \gamma_0$ (Theorem 2.6.8).

2.6.1 Preliminaries

First we give definitions and preliminaries of viscosity solutions to the free boundary problem and discuss the half-plane solution.

A point $X \in \mathbb{R}^{n+1}$ is denoted by $X = (x, y) \in \mathbb{R}^n \times \mathbb{R}$. We use the notation $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$. For a function g defined in $(B_1^{n+1})^+ = \{X \in \mathbb{R}^{n+1}, |X| < 1, y > 0\}$, let $\Omega^+(g) = \{g(x,0) > 0\} \cap B_1^n$ denote the positive set in $\mathbb{R}^n$, and let $F(g) = \partial_{\mathbb{R}^n}\Omega^+(g) \cap B_1^n$ denote the free boundary. Let $\mathcal{G}(u) = \partial\{u > 0\} \cap \partial B_1^n \subset \partial B_1^{n+1}$ denote the boundary of the set $\partial\{u > 0\} \cap \partial B_1^n$ in ∂B_1^{n+1}. We consider the free boundary problem

$$\begin{cases} div(y^\alpha \nabla g) = 0 \quad in \quad (B_1^{n+1})^+, \\ \frac{\partial g}{\partial U} = 1 \quad on \quad F(g), \\ \lim_{y\to 0^+} y^\alpha \partial_y g(x,y) = \gamma g^{\gamma-1}(x) \quad in \quad \Omega^+(g), \end{cases} \tag{2.6.1}$$

where

$$\frac{\partial g}{\partial U}(x) = \lim_{t\to 0^+} \frac{g(x + t\nu(x), 0)}{t^\beta}, x \in F(g),$$

and $\nu(x)$ is the unit normal to $F(g)$ at x towards the positive set $\Omega^+(g)$, and $U = U_\gamma$ is defined as the following.

Consider $U(t,z)$ as the extension of $(t)_+^\beta$ to upper half plane, which satisfies $U(t,0) = (t)_+^\beta$, and $div(z^\alpha \nabla U(t,z)) = 0$ in $\{t \in \mathbb{R}, z > 0\}$.
Write $U(t,z) = r^\beta g(\theta)$, $r = \sqrt{t^2 + z^2} > 0$, $t = r\cos\theta$, $z = r\sin\theta$, and $\theta \in [0,\pi]$. Then the equation for $g(\theta) \geq 0$ is

$$g''(\theta) + \alpha \cot\theta g'(\theta) + \beta(\alpha+\beta)g(\theta) = 0$$

with $g(\pi) = 0, g(0) = 1$, and $g(\theta) = 1 + \gamma(\sin\theta)^{2s} + o((\sin\theta)^{2s})$. The last equation is derived from the equation $\lim_{z\to 0} z^\alpha \partial_z U(t,z) = \gamma U^{\gamma-1}(t,0)$ when $t > 0$. The $(n+1)$-dimensional function $U(X) = U(x_n, z)$ is a solution of (2.6.1) with free boundary $\{x_n = 0\}$.

2.6.2 Viscosity solutions

We now introduce the definition of viscosity solutions to (2.6.1).

Definition 2.6.1. Given g, v continuous, we say that v touches g by below (resp. above) at X_0 if $g(X_0) = v(X_0)$ and

$$g(X) \geq v(X) \quad \text{(resp. } g(X) \leq v(X)\text{) in a neighborhood } O \text{ of } X_0.$$

If this inequality is strict in $O \setminus \{X_0\}$, we say that v touches g strictly by below (resp. above).

Definition 2.6.2. We say $v \in C((B_1^{n+1})^+)$ is a (strict) comparison subsolution to (2.6.1) if v is a non-negative function in $(B_1^{n+1})^+$ which is C^2 in the set where it is positive, and it satsfies

(i) $div(y^\alpha \nabla v) \geq 0 \;\; in \;\; (B_1^{n+1})^+$.

(ii) $F(v)$ is C^2 and if $x_0 \in F(v)$ we have

$$v(x,y) = aU((x-x_0)\cdot\nu(x_0), y) + o(|(x-x_0,y)|^\beta), \quad \text{as } (x,y) \to (x_0, 0),$$

with $a \geq 1$, and $\nu(x_0)$ denotes the unit normal at x_0 to $F(v)$ towards the positive set $\Omega^+(v)$.

(iii) $\lim_{y\to 0^+} y^\alpha \partial_y v(x,y) \geq \gamma v^{\gamma-1}(x)$.

(iv) Either v satisfies (i) and (iii) strictly or $a > 1$.

Similarly one can define a comparison supersolution.

Definition 2.6.3. We say that g is a viscosity solution to (2.6.1) if g is a continuous non-negative function which satisfies

(i) g is locally $C^{1,1}$ in $(B_1^{n+1})^+$ and solves (in the viscosity sense)

$$\begin{cases} div(y^\alpha \nabla g) = 0 \quad in \quad (B_1^{n+1})^+, \\ \lim_{y\to 0^+} y^\alpha \partial_y g(x,y) = \gamma g^{\gamma-1}(x,0) \quad in \quad \Omega^+(g). \end{cases}$$

(ii) Any (strict) comparison subsolution (resp. supersolution) cannot touch g by below (resp. by above) at a point $X_0 = (x_0, 0) \in F(g)$.

2.6.3 Energy minimizers are viscosity solutions

In this part, we prove when γ is small enough, if g is a minimizer of the energy functional J_γ, then it is a viscosity solution of the equations (2.6.5).

Lemma 2.6.4. *There exists $\gamma_0 > 0$ such that for each $0 < \gamma < \gamma_0$, if $g = g_\gamma$ satisfies the following conditions: $g_\gamma \in C^\beta((B_1^{n+1})^+)$, $g \geq 0$ and g solves*

$$\begin{cases} div(y^\alpha \nabla g) = 0 \quad in \quad (B_1^{n+1})^+, \\ \lim_{y\to 0^+} y^\alpha \partial_y g(x,y) = \gamma g^{\gamma-1}(x,0) \quad in \quad \Omega^+(g), \end{cases}$$

with $(0,0)$ *a free boundary point and* $B^n_{1/2}(1/2e_n) \subset \{x \in B^n_1, g(x,0) > 0\}$, *then*

$$g(X) = T_\gamma U_\gamma(x_n, y) + o(|X|^\beta).$$

for some $T_\gamma > 0$, *with* U_γ *defined in Scetion 6.1.*

Proof. Let

$$T_\gamma = \inf_{\nu \notin P} \liminf_{t \to 0^+} \frac{g_\gamma}{U_\gamma}(t\nu),$$

with $P = \{(x', x_n, 0), x_n < 0\}$ the half hyperplane. We can see $T_\gamma > 0$ uniformly as $\gamma \to 0$ by the non-degeneracy property of g_γ in Section 2.5. Assume by contradiction that the conclusion of the lemma does not hold with this choice of T_γ. Then there exists $\delta_1 > 0$ and a sequence of points $Y_k \to 0$ such that

$$|g_\gamma(Y_k) - T_\gamma U_\gamma(Y_k)| \geq \delta_1 |Y_k|^\beta. \tag{2.6.2}$$

Since g_γ is C^β Hölder continuous, the rescalings

$$g_{k,\gamma}(X) = |Y_k|^{-\beta} g_\gamma(|Y_k| X)$$

are uniformly C^β continuous. Assume that as $k \to \infty$, $g_{k,\gamma} \to g_{*,\gamma}$ uniformly on compact sets after passing to a subsequence. Then we obtain $g_{*,\gamma} \geq T_\gamma U_\gamma$ by the definition of T_γ and

$$\begin{cases} div(y^\alpha \nabla g_{*,\gamma}) = 0 \quad in \quad (B^{n+1}_1)^+, \\ \lim_{y \to 0^+} y^\alpha \partial_y g_{*,\gamma}(x,y) = \gamma g^{\gamma-1}_{*,\gamma}(x) \quad in \quad \Omega^+(g_{*,\gamma}) \supset \{x_n \geq 0\} \cap B^n_1. \end{cases}$$

In view of (2.6.2), there exists a point Y_*, $|Y_*| = 1$ such that

$$g_{*,\gamma}(Y_*) \geq T_\gamma U_\gamma(Y_*) + \delta_1.$$

By Lemma 2.8.4,

$$g_{*,\gamma} \geq (1+\delta_2)T_\gamma U_\gamma$$

for some $\delta_2 > 0$ small. Therefore, for all large k,

$$g_{k,\gamma} \geq (1+\delta_2)T_\gamma U_\gamma - \epsilon_k \quad in \quad (B_1^{n+1})^+ \tag{2.6.3}$$

for some $\epsilon_k \to 0$. Then our aim is to prove when γ is small enough, there exists some $C > 0$ such that

$$g_{k,\gamma} \geq (1+\delta_2)T_\gamma(1 - C\frac{\epsilon_k}{T_\gamma})U_\gamma \quad in \quad (B_{1/2}^{n+1})^+.$$

We prove by the method of compactness. Assume by contradiction the result does not hold. Fixing k large enough, there exists $\gamma_j \to 0$, and $Z_j \in (B_{1/2}^{n+1})^+$, such that for all $C_j > 0$ and $1 - C_j\frac{\epsilon_k}{T_{\gamma_j}} > 0$,

$$g_{k,\gamma_j}(Z_j) < T_{\gamma_j}(1+\delta_2)(1 - C_j\frac{\epsilon_k}{T_{\gamma_j}})U_{\gamma_j}(Z_j).$$

Let $\gamma_j \to 0$, and take $Z_\infty \in (B_{1/2}^{n+1})^+$ as the limit of a subseuqence of Z_j. The function $g_{k,0} = \lim_{\gamma_j \to 0} g_{k,\gamma_j}$ satisfies

$$g_{k,0}(Z_\infty) \leq T_0(1+\delta_2)(1 - C'\frac{\epsilon_k}{T_0})U_0(Z_\infty) \tag{2.6.4}$$

for all $C' > 0$ and $1 - C'\frac{\epsilon_k}{T_0} > 0$. However, as $\gamma_j \to 0$, (2.6.3) shows

$$g_{k,0} \geq T_0(1+\delta_2)U_0 - \epsilon_k \quad in \quad (B_1^{n+1})^+,$$

and by the proof of Lemma 7.5 in [17], it leads to

$$g_{k,0} \geq T_0(1+\delta_2)(1 - C\frac{\epsilon_k}{T_0})U_0 \quad in \quad (B_{1/2}^{n+1})^+,$$

which leads to a contradiction of (2.6.4) here.

Therefore, when γ is small enough, for ϵ_k small enough,

$$g_{k,\gamma} \geq (1+\delta_2)T_\gamma(1 - C\frac{\epsilon_k}{T_\gamma})U_\gamma \geq (1+\delta_2/2)T_\gamma U_\gamma \quad in \quad (B_{1/2}^{n+1})^+.$$

This implies for any $\nu \notin P$,

$$\liminf_{t\to 0^+} \frac{g_\gamma}{U_\gamma}(t\nu) = \liminf_{t\to 0^+} \frac{g_{k,\gamma}}{U_\gamma}(t\nu) \geq T_\gamma(1+\delta_2/2)$$

which contradicts the minimality of T_γ. Then it leads to the conclusion that there exists $\gamma_0 > 0$ such that for each $0 < \gamma < \gamma_0$,

$$g_\gamma = T_\gamma U_\gamma + o(|X|^\beta).$$

□

This lemma leads to the following proposition.

Proposition 2.6.5. *There exists $\gamma_0 > 0$ such that for each $0 < \gamma < \gamma_0$, if g is a minimizer of energy functional J_γ, then g is a viscosity solution of the following equations:*

$$\begin{cases} div(y^\alpha \nabla g) = 0 \quad in \quad (B_1^{n+1})^+, \\ \frac{\partial g}{\partial U_\gamma} = A(s,\gamma) \quad on \quad F(g), \\ \lim_{y\to 0^+} y^\alpha \partial_y g(x,y) = \gamma g^{\gamma-1}(x) \quad in \quad \Omega^+(g), \end{cases} \tag{2.6.5}$$

with $A = A(s,\gamma)$ defined in (2.4.1).

Proof. We only need to check if the minimizer g satisfies the free boundary condition $\frac{\partial g}{\partial U_\gamma} = A(s,\gamma)$.

Assume we touch $F(g)$ at 0 with $B^n_\delta(\delta e_n)$ from the positive set $\{g > 0\}$. By Lemma 2.6.4, g has an expansion

$$g(X) = T_\gamma U_\gamma(x_n, y) + o(|X|^\beta)$$

with $T_\gamma > 0$. The rescaled solutions $\lambda^{-\beta} g(\lambda X) \to T_\gamma U_\gamma$ uniformly, and by Corollary 2.2.2, $T_\gamma U_\gamma$ is a global minimizer in $(\mathbb{R}^2)^+$. Then as proved in Section 2.4, Theorem 2.4.1, $T_\gamma = A(s, \gamma)$. This shows the minimizer g satisfies the free boundary condition

$$\frac{\partial g}{\partial U_\gamma} = A(s, \gamma),$$

and g is a viscosity solution of (2.6.5).

□

For simplicity, we work on the viscosity solution of (2.6.1) with constant $A(s, \gamma)$ replaced by 1.

2.6.4 Comparison principle

We state the comparison principle for the problem (2.6.1). The proof is standard and can be found at Lemma 2.6 in [16].

Lemma 2.6.6. *Let $g, v_t \in C(\overline{(B_1)^+})$ be respectively a solution and a family of subsolutions to* (2.6.1) *with $0 \le t \le 1$. Assume that*

(i) $v_0 \le g$ *in* $\overline{(B_1)^+}$.

(ii) $v_t \leq g$ *on* $(\partial B_1^{n+1})^+$ *for all* $t \in [0,1]$.

(iii) $v_t < g$ *on* $\mathcal{G}(v_t) = \partial\{v_t > 0\} \cap \partial B_1^n \subset \partial B_1^{n+1}$.

(iv) $v_t(x)$ *is continuous in* $(x,t) \in \overline{(B_1)^+} \times [0,1]$ *and* $\overline{\{v_t > 0\} \cap B_1^n}$ *is continunous in the Hausdorff metric.*

Then

$$v_t \leq g \quad \text{in} \quad \overline{(B_1)^+} \quad \text{for all} \quad t \in [0,1].$$

As a consequence of the lemma, we introduce the comparison principle used in this paper.

Corollary 2.6.7. *Let* g *be a solution to* (2.6.1) *and let* v *be a subsolution to* (2.6.1) *in* $(B_2^{n+1})^+$ *which is strictly monotone in the* e_n*-direction in the set* $\{v > 0\} \cap B_2^{n+1} \cap \{y \geq 0\}$. *Call*

$$v_t(X) = v(X + te_n), X \in B_1^+.$$

Assume that for $-1 \leq t_0 \leq t_1 \leq 1$,

$$v_{t_0} \leq g \quad \text{in} \quad \overline{(B_1^{n+1})^+},$$

and

$$v_{t_1} \leq g \quad \text{on} \quad \partial(B_1^{n+1})^+, \quad v_{t_1} < g \quad \text{on} \quad \mathcal{G}(v_{t_1}).$$

Then

$$v_{t_1} \leq g \quad \text{in} \quad \overline{(B_1^{n+1})^+}.$$

2.6.5 Main theorem

Theorem 2.6.8. *There exists $\gamma_0 > 0$ such that for each $0 < \gamma < \gamma_0$, there exists a universal constant $\bar{\epsilon} > 0$, such that if g is a viscosity solution to* (2.6.1) *satisfying the flatness condition*

$$\{x \in B_1^n, x_n \leq -\bar{\epsilon}\} \subset \{x \in B_1^n, g(x,0) = 0\} \subset \{x \in B_1^n, x_n \leq \bar{\epsilon}\},$$

then $F(g)$ is $C^{1,\theta}$ in $B_{1/2}^n$, with $\theta > 0$ depending on n, s and γ.

Lemma 2.6.9. *Assume g_γ solves* (2.6.1), *and U_γ is the half-plane solution. There exists $\gamma_0 > 0$ such that for each $0 < \gamma < \gamma_0$, given any $\epsilon > 0$, there exists $\bar{\epsilon} > 0$ and $\delta > 0$ depending on ϵ such that if*

$$\{x \in B_1^n, x_n \leq -\bar{\epsilon}\} \subset \{x \in B_1^n, g_\gamma(x,0) = 0\} \subset \{x \in B_1^n, x_n \leq \bar{\epsilon}\},$$

then the rescaling $\delta^{-\beta} g_\gamma(\delta X)$ satisfies

$$U_\gamma(X - \epsilon e_n) \leq \delta^{-\beta} g_\gamma(\delta X) \leq U_\gamma(X + \epsilon e_n) \quad \text{in} \quad (B_1^{n+1})^+.$$

Proof. We use the method of compactness since this lemma for the case $\gamma = 0$ is proved in Lemma 2.10 in [13]. Assume that there exists $\gamma_k \to 0$ such that the lemma does not hold for each γ_k. Then for each γ_k, there exists a sequence $\{g_{\gamma_k}^j\}_{j=1}^\infty$, $g_{\gamma_k}^j$ are solutions of (2.6.1) with $\gamma = \gamma_k$, and a sequence $\{\bar{\epsilon}_k^j\}_{j=1}^\infty$ with $\bar{\epsilon}_k^j \to 0$ as $j \to \infty$ for each k, such that $g_{\gamma_k}^j$ satisfies the following condition with $\bar{\epsilon}_k^j \to 0$ as $j \to \infty$,

$$\{x \in B_1^n, x_n \leq -\bar{\epsilon}_k^j\} \subset \{x \in B_1^n, g_{\gamma_k}^j(x,0) = 0\} \subset \{x \in B_1^n, x_n \leq \bar{\epsilon}_k^j\},$$

but the conclusion does not hold for $\delta_k^j \to 0$ as $j \to \infty$.

Let $g_0^j = \lim_{\gamma_k \to 0} g_{\gamma_k}^j$, the limit exists since in [20] the optimal C^β estimates for the minimizers are given, with $\beta = \frac{2s}{2-\gamma} > s$ and the C^β norm does not blow-up as $\gamma \to 0$. Let $\bar{\epsilon}_0^j = \lim_{k\to\infty} \bar{\epsilon}_k^j \to 0$ as $j \to \infty$. The limit $U_0(X) = \lim_{\gamma_k \to 0} U_{\gamma_k}$ is the half-plane solution for the one-phase cavitation problem. In addition, in Lemma 2.5.2 the minimizers are uniformly non-degenerate as $\gamma \to 0$. Then $\{u_0^j\}_{j=1}^\infty$ are the solutions of the case $\gamma = 0$, and satisfy the flatness assumption with width $\sup_k \bar{\epsilon}_{\gamma_k}^j \to 0$ as $j \to \infty$, but the conclusion does not hold, which leads to a contradiction. $\square$

So from now on we assume that

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \ \text{ in } \ (B_1^{n+1})^+.$$

The proof of Theorem 2.6.8 is organized as follows. In Section 2.7 we recall the $\epsilon-$domain variation of the solutions and the associated linearized equations. In Section 2.8 we give the proof of a Harnack inequality and then we improve the flatness in Section 2.9. In Section 2.10 the regularity of the solutions to the linearized equations is proved and we finish our proof of Theorem 2.6.8 in Section 2.11. In the Appendix, several useful inequalities of the half-plane solution $U(t, z)$ are given.

2.7 The linearized problem

In this section we recall the $\epsilon-$domain variation of the solution to (2.6.1) and state the associated linearized problem, which is introduced in [13].

2.7.1 The ϵ-domain variations

Let $P = \{X \in \mathbb{R}^{n+1}, x_n \leq 0, y = 0\}$ and $L = \{X \in \mathbb{R}^{n+1}, x_n = 0, y = 0\}$. To each $X \in \mathbb{R}^{n+1} \cap \{y \geq 0\} \setminus P$ we associate a set $\tilde{g}_\epsilon(X) \subset \mathbb{R}$ such that

$$U(X) = g(X - \epsilon w e_n), \quad \forall w \in \tilde{g}_\epsilon(X).$$

We call $\tilde{g}_\epsilon$ the ϵ-domain variation associated to g. From now on we let $\tilde{g}_\epsilon(X)$ denote any of the values in this set, by abuse of notation. We claim the following: if g satisfies

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \ \text{ in } \ B_\rho^{n+1} \cap \{y \geq 0\}, \tag{2.7.1}$$

then

$$\tilde{g}_\epsilon(X) \in [-1, 1].$$

To prove this, same as in [16], we let

$$Y = X - \epsilon \tilde{g}_\epsilon(X) e_n, \quad X \in \mathbb{R}^{n+1} \cap \{y \geq 0\} \setminus P,$$

then we can see

$$U(Y - \epsilon e_n) \leq g(Y) = U(Y + \epsilon \tilde{g}_\epsilon(X) e_n) \leq U(Y + \epsilon e_n),$$

by our definition $U(X) = g(X - \epsilon \tilde{g}_\epsilon(X) e_n) > 0$ and U is strictly monotone in e_n-direction outside of P. By (2.7.1), for each $X \in B_{\rho-\epsilon}^{n+1} \cap \{y \geq 0\} \setminus P$, the set $\tilde{g}_\epsilon(X)$ is non-empty and there exists at least one value such that

$$U(X) = g(X - \epsilon \tilde{g}_\epsilon(X) e_n).$$

Our claim follows by the continuity of $g(X - \delta\epsilon e_n)$, for $\delta \in [-1, 1]$.

Moreover, if g is strictly monotone in the e_n-direction, the $\tilde{g}_\epsilon(X)$ is single-valued.

The following lemma is useful to obtain a comparison principle.

Lemma 2.7.1. *Let g, v be respectively a solution and a subsolution to* (2.6.1) *in $(B_2^{n+1})^+$. Assume that g satisfies the flatness condition* (2.7.1) *in $(B_2^{n+1})^+$, v is strictly increasing in the e_n-direction in $\{v > 0\} \cap B_\rho^{n+1} \cap \{y \geq 0\}$, and $\tilde{v}_\epsilon$ is defined on $B_{2-\epsilon}^{n+1} \cap \{y \geq 0\} \setminus P$ with*

$$|\tilde{v}_\epsilon| \leq C < \infty.$$

If

$$\tilde{v}_\epsilon + c \leq \tilde{g}_\epsilon \quad \text{in} \quad B_{3/2}^{n+1} \setminus \overline{B_{1/2}^{n+1}} \cap \{y \geq 0\} \setminus P,$$

then we have

$$\tilde{v}_\epsilon + c \leq \tilde{g}_\epsilon \quad \text{on} \quad B_{3/2}^{n+1} \cap \{y \geq 0\} \setminus P.$$

The proof given in Lemma 3.2 in [16] is still valid since it only involves the comparison principle in Corollary 2.6.7 and the definition of $\tilde{g}_\epsilon$.

Given $\epsilon > 0$ and a Lipschitz function $\tilde{\psi}$ defined on $B_\rho^{n+1}(Y) \cap \{y \geq 0\}$ with values in $[-1, 1]$, there exists a unique function ψ_ϵ defined on $B_{\rho-\epsilon}^{n+1}(Y) \cap \{y \geq 0\}$ such that

$$U(X) = \psi_\epsilon(X - \epsilon\tilde{\psi}(X)e_n), X \in B_\rho^{n+1}(Y) \cap \{y \geq 0\}.$$

Moreover ψ_ϵ is increasing in the e_n direction. Thus, if g satisfies the flatness condition (2.7.1) and $\tilde{\psi}$ is defined as above, then

$$\tilde{\psi} \leq \tilde{g}_\epsilon \;\; \text{in} \;\; B_\rho^{n+1}(Y) \cap \{y \geq 0\} \setminus P$$

leads to

$$\psi_\epsilon \leq g \;\; \text{in} \;\; B_{\rho-\epsilon}^{n+1}(Y) \cap \{y \geq 0\}. \tag{2.7.2}$$

2.7.2 The linearized problem

We introduce here the linearized problem associated to (2.6.1). U_n is the x_n-derivative of the function U. Given $w \in C((B_1^{n+1})^+)$ and $X_0 = (x_0', 0, 0)$, we define

$$|\nabla_r w|(X_0) = \lim_{(x_n,y)\to(0,0)} \frac{w(x_0', x_n, y) - w(x_0', 0, 0)}{r}, r^2 = x_n^2 + y^2.$$

The linearized problem associated to (2.6.1) is

$$\begin{cases} div(y^\alpha \nabla(U_n w)) = 0 \;\; in \;\; (B_1^{n+1})^+, \\ |\nabla_r w|(X_0) = 0 \;\; \text{on} \;\; B_1^n \cap L, \\ \lim_{y\to 0^+} y^\alpha \partial_y w(x,y) = 0 \;\; \text{on} \;\; B_1^n \cap \{x_n > 0\}. \end{cases} \tag{2.7.3}$$

The definition of the viscosity solution for this problem is the following.

Definition 2.7.2. We say that w is a solution to (2.7.3) if $w \in C_{loc}^{1,1}((B_1^{n+1})^+)$ and it satisfies (in the viscosity sense)

(i)
$$\begin{cases} div(y^\alpha \nabla(U_n w)) = 0 \;\; in \;\; (B_1^{n+1})^+, \\ \lim_{y\to 0^+} y^\alpha \partial_y w(x,y) = 0 \;\; \text{on} \;\; B_1^n \cap \{x_n > 0\}. \end{cases}$$

(ii) Let ϕ be continuous around $X_0 = (x_0', 0, 0) \in B_1^n \cap L$ and satisfies

$$\phi(X) = \phi(X_0) + a(X_0) \cdot (x' - x_0') + b(X_0)r + O(|x' - x_0'|^2 + r^{1+\theta}),$$

for some $\theta > 0$ and $b(X_0) \neq 0$.

If $b(X_0) > 0$ then ϕ cannot touch w by below at X_0, and if $b(X_0) < 0$ then ϕ cannot touch w by above at X_0.

2.8 Harnack inequality

In this section, we prove the following Harnack type inequality for solutions to the free boundary problem (2.6.1).

Theorem 2.8.1 (Harnack inequality)**.** *There exists $\bar{\epsilon} > 0$ such that if g solves* (2.6.1) *and it satisfies*

$$U(X + \epsilon a_0 e_n) \leq g(X) \leq U(X + \epsilon b_0 e_n) \quad \text{in} \quad (B_\rho^{n+1}(X^*))^+,$$

with $\epsilon(b_0 - a_0) \leq \bar{\epsilon}\rho$, then

$$U(X + \epsilon a_1 e_n) \leq g(X) \leq U(X + \epsilon b_1 e_n) \quad \text{in} \quad (B_{\eta\rho}^{n+1}(X^*))^+,$$

with

$$a_0 \leq a_1 \leq b_1 \leq b_0, \quad b_1 - a_1 \leq (1 - \eta)(b_0 - a_0),$$

for a small universal constant η.

Let g be a solution to (2.6.1) which satisfies

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \;\text{ in }\; (B_1^{n+1})^+.$$

Let A_ϵ be the set

$$A_\epsilon = \{(X, \tilde{g}_\epsilon(X)) : X \in (B^{n+1}_{1-\epsilon})^+\} \subset \mathbb{R}^{n+1} \times [a_0, b_0].$$

Since $\tilde{g}_\epsilon$ may be multi-valued, $(X, \tilde{g}_\epsilon(X))$ denotes all possible values of $\tilde{g}_\epsilon$. An iterative argument gives the following corollary of Theorem 2.8.1.

Corollary 2.8.2. *If*

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad in \quad (B_1^{n+1})^+$$

with $\epsilon \leq \bar{\epsilon}/2$*, given* $m_0 > 0$ *such that*

$$2\epsilon(1-\eta)^{m_0}\eta^{-m_0} \leq \bar{\epsilon},$$

then the set $A_\epsilon \cap ((B^{n+1}_{1/2})^+ \times [-1, 1])$ *is above the graph of a function* $z = a_\epsilon(X)$ *and is below the graph of a function* $z = b_\epsilon(X)$ *with*

$$b_\epsilon - a_\epsilon \leq 2(1-\eta)^{m_0 - 1},$$

and a_ϵ, b_ϵ *having a modulus of continuity bounded by the Hölder function* At^B *with* A, B *depending only on* η.

The proof of the Harnack inequality follows as in the case when $\gamma = 0$ in [13]. The key ingredient is the lemma below.

Lemma 2.8.3. *There exists* $\bar{\epsilon} > 0$ *such that for all* $0 < \epsilon < \bar{\epsilon}$*, if* g *is a solution to* (2.6.1) *such that*

$$g(X) \geq U(X) \quad in \quad (B^{n+1}_{1/2})^+,$$

and at $\bar{X} \in (B^{n+1}_{1/8}(\frac{1}{4}e_n))^+$

$$g(\bar{X}) \geq U(\bar{X} + \epsilon e_n), \tag{2.8.1}$$

then

$$g(X) \geq U(X + \tau\epsilon e_n) \quad in \quad (B^{n+1}_{\delta})^+ \tag{2.8.2}$$

for universal constants τ, δ*. Similarly, if*

$$g(X) \leq U(X) \quad in \quad (B^{n+1}_{1/2})^+,$$

and

$$g(\bar{X}) \leq U(\bar{X} - \epsilon e_n),$$

then

$$g(X) \leq U(X - \tau\epsilon e_n) \quad in \quad (B^{n+1}_{\delta})^+.$$

There is a preliminary lemma.

Lemma 2.8.4. *Let* $g \geq 0$ *be* $C^{1,1}_{loc}$ *in* $(B^{n+1}_2)^+$ *and solves*

$$\begin{cases} div(y^{\alpha}\nabla g) = 0 \quad in \quad (B^{n+1}_2)^+, \\ \lim_{y \to 0} y^{\alpha}\partial_y g = \gamma g^{\gamma-1} \quad on \quad \{g > 0\} \cap B^n_2. \end{cases}$$

Let $\bar{X} = \frac{3}{2}e_n$*. Assume that*

$$g \geq U \quad in \quad (B^{n+1}_2)^+, \quad g(\bar{X}) - U(\bar{X}) \geq \delta_0$$

for some $\delta_0 > 0$*. Then*

$$g \geq (1 + c\delta_0)U \quad in \quad (B^{n+1}_1)^+,$$

for a small universal constant c*. In particular, for any* $0 < \epsilon < 2$*,*

$$U(X + \epsilon e_n) \geq (1 + c\epsilon)U(X) \quad in \quad (B^{n+1}_1)^+.$$

The proof is slightly different since the boundary Harnack inequality of U does not work. So instead we have the following proof.

Proof. We do an even extension of U and g with resepect to $\{y=0\}$, and let g^*-U solves the following equation:

$$\begin{cases} div(y^\alpha \nabla(g^*-U))=0 \;\text{ in }\; D=(B_{3/2}^{n+1})\setminus\{x_n<0,y=0\}, \\ g^*-U=g-U\geq 0 \;\text{ on }\; \partial B_{3/2}^{n+1}, \\ g^*-U=0 \;\text{ on }\; \{x_n<0,y=0\}. \end{cases}$$

Then g^* satisfies

$$\begin{cases} div(y^\alpha \nabla g^*)=0 \;\text{ in }\; (B_{3/2}^{n+1})^+, \\ g^*=g \;\text{ on }\; (\partial B_{3/2}^{n+1})^+, \\ g^*=0\leq g \;\text{ on }\; \{x_n<0,y=0\}, \\ \lim_{y\to 0} y^\alpha \partial_y g^* \geq \lim_{y\to 0} y^\alpha \partial_y g \;\text{ on }\; \{x_n>0,y=0\}. \end{cases}$$

The last inequality holds since

$$\lim_{y\to 0} y^\alpha \partial_y g^* = \lim_{y\to 0} y^\alpha \partial_y U = \gamma U^{\gamma-1} \geq \gamma g^{\gamma-1} = \lim_{y\to 0} y^\alpha \partial_y g.$$

By maximum principle, $g^*\leq g$ in $(B_{3/2}^{n+1})^+$. Let $\bar{X}=\frac{3}{2}e_n$, and $g(\bar{X})-U(\bar{X})\geq \delta_0$. Since g^*-U satisfies the Harnack inequality, we can see

$$g^*-U=g-U\geq c_0\delta_0 \;\text{ on }\; (\partial B_{3/2}^{n+1})^+\cap B_{1/4}^{n+1}(\bar{X}),$$

and

$$g^*(\tilde{X})-U(\tilde{X})\geq C_1\delta_0$$

at some $\tilde{X}\in B_1^{n+1}\cap D$. Since g^*-U satisfies the boundary Harnack inequality,

$$g^*(X)-U(X)\geq C_2\frac{g^*(\tilde{X})-U(\tilde{X})}{V(\tilde{X})}V(X) \;\text{ in }\; (B_1^{n+1})^+.$$

Here $V(X)$ solves

$$div(y^\alpha \nabla V) = 0 \ \text{ in } \ D,$$

and $V(X) = 0$ on $\{x_n < 0, y = 0\}$. We can let $V(X)$ be the function defined in Section 2.2 in [13], which is the extension of $(x_n)_+^s$. Here we want to prove

$$V(X) \geq CU(X) \ \text{ in } \ (B_1^{n+1})^+. \tag{2.8.3}$$

In the 2-dimensional case, let $X = (x, y) \in (\mathbb{R}^2)^+$, $x = |X| \cos\theta$ and $y = |X| \sin\theta$ for $\theta \in [0, \pi]$. Using the homogenity property of U and V, we can see

$$\begin{cases} V(X) = |X|^s V(\frac{X}{|X|}) = |X|^s h(\theta), \\ U(X) = |X|^\beta U(\frac{X}{|X|}) = |X|^\beta f(\theta). \end{cases}$$

and $\beta = \frac{2s}{2-\gamma} > s$. So we want to prove $\frac{h(\theta)}{f(\theta)} \geq C > 0$ for $\theta \in [0, \pi]$. From Section 2.2 in [13], $h(\theta) = (cos(\theta/2))^{2s}$. From Section 2.6, $f(\theta) \geq 0$ solves the ODE

$$f''(\theta) + \alpha \cot\theta f'(\theta) + \beta(\alpha + \beta) f(\theta) = 0 \tag{2.8.4}$$

with $f(\pi) = 0, f(0) = 1$, and $f(\theta) = 1 + \gamma(sin\theta)^{2s} + o((sin\theta)^{2s})$ as $\theta \to 0$. So we only need to consider the case near $\theta = \pi$, where $h(\pi) = f(\pi) = 0$. We can see

$$\begin{aligned} \lim_{\theta\to\pi} \frac{h(\theta)}{f(\theta)} &= \lim_{\theta\to\pi} \frac{cos(\theta/2)^{2s}}{f(\theta)} \\ &= \lim_{\theta\to\pi} \frac{s(cos(\theta/2))^{2s-1}(-sin(\theta/2))}{f'(\theta)} \\ &= \lim_{\theta\to\pi} \frac{(-s)cos(\theta/2)^{2s-1} sin(\theta/2) sin(\theta)^{1-2s}}{f'(\theta)(sin\theta)^\alpha} \\ &= \lim_{\theta\to\pi} \frac{(-s)2^{1-2s}(sin(\theta/2))^{2-2s}}{f'(\theta)(sin\theta)^\alpha}. \end{aligned}$$

Our aim is to prove

$$\gamma \geq f'(\theta)(sin\theta)^{\alpha} \geq \gamma - C_0\beta(\alpha+\beta)\|f\|_{L^\infty}. \tag{2.8.5}$$

Since

$$\begin{aligned}\lim_{\theta\to 0} f'(\theta)(\sin\theta)^{\alpha} &= \lim_{\theta\to 0}\frac{f(\theta)-f(0)}{\theta}(\sin\theta)^{\alpha}\\ &= \lim_{\theta\to 0}\frac{\gamma\sin\theta^{2s}}{\theta}(\sin\theta)^{\alpha}\\ &= \gamma,\end{aligned}$$

and f solves the equation (2.8.4), which is equivalent to

$$(f'(\theta)(sin\theta)^{\alpha})' = -\beta(\alpha+\beta)(\sin\theta)^{\alpha}f(\theta), \tag{2.8.6}$$

we can apply fundamental theorem of calculus and get

$$f'(\theta)(\sin\theta)^{\alpha} = \gamma - \beta(\alpha+\beta)\int_0^{\theta}(\sin\phi)^{\alpha}f(\phi)d\phi,$$

so we need to prove $C_0 = \int_0^{\pi}(\sin\theta)^{1-2s}d\theta > 0$ is a bounded number, which is ensured since $1-2s > -1$. Now it is confirmed that in $[0,\pi]$,

$$\gamma \geq f'(\theta)(sin\theta)^{\alpha} \geq \gamma - C.$$

So if $\tilde{C} \leq f'(\theta)(sin\theta)^{\alpha} \leq 0$ for some $\tilde{C} \leq 0$, then the limit is a positive number (including positive infinity) and we complete the proof of (2.8.3). If not, then it leads to a contradiction, since $\frac{h(\theta)}{f(\theta)} \geq 0$.

From above we prove that

$$V(X) \geq CU(X) \ \text{ in } \ (B_1^{n+1})^+.$$

Then the proof follows as

$$g^*(X) - U(X) \geq C_2 \frac{g^*(\tilde{X}) - U(\tilde{X})}{V(\tilde{X})} V(X) \geq C\delta_0 U(X) \text{ in } (B_1^{n+1})^+,$$

and

$$g(X) \geq g^*(X) \geq (1 + C\delta_0)U(X).$$

$\square$

In the proof of Lemma 2.8.3, we use the following family of radial subsolutions. Let $R > 0$ and define

$$V_R(t,z) = U(t,z)((n-1)\frac{t}{R} + 1).$$

Then set the (n+1)-dimensional function v_R by rotating fuction V_R around $(0, R, z)$,

$$v_R(X) = V_R(R - \sqrt{|x'|^2 + (x_n - R)^2}, z). \tag{2.8.7}$$

Proposition 2.8.5. *If R is large enough, the function v_R is a comparison subsolution to* (2.6.1) *in $(B_2^{n+1})^+$ which is strictly monotone increasing in the e_n-direction. Moreover, there exists a function $\tilde{v}_R$ such that*

$$U(X) = v_R(X - \tilde{v}_R(X)e_n) \quad in \quad (B_1^{n+1})^+ \tag{2.8.8}$$

and

$$|\tilde{v}_R(X) - \gamma_R(X)| \leq \frac{C}{R^2}|X|^2, \gamma_R(X) = -\frac{|x'|^2}{2R} + 2(n-1)\frac{x_n r}{R}, \tag{2.8.9}$$

with $r = \sqrt{x_n^2 + z^2}$ and C universal.

Proof. Step 1. In this part we prove that v_R is a comparison subsolution and is strictly monotone increasing in the e_n-direction.

First, we need to prove v_R is a strict subsolution to

$$div(z^\alpha \nabla v_R) = 0 \text{ in } (B_2^{n+1})^+. \tag{2.8.10}$$

We can compute that

$$\begin{aligned}
&\Delta v_R + \frac{\alpha}{z}\partial_z v_R \\
&\quad = \Delta_{t,z} V_R(R-\rho, z) - \frac{n-1}{\rho}\partial_t V_R(R-\rho, z) + \frac{\alpha}{z}\partial_z V_R(R-\rho, z),
\end{aligned}$$

where $\rho = \sqrt{|x'|^2 + (x_n - R)^2}$. Then for $(t, z) \in (\mathbb{R}^2)^+$,

$$\begin{aligned}
\Delta_{t,z} v_R(t, z) &+ \frac{\alpha}{z}\partial_z V_R(t, z) \\
&= (\partial_{tt} + \partial_{zz})V_R(t, z) + \frac{\alpha}{z}\partial_z V_R(t, z) \\
&= \frac{2(n-1)}{R}\partial_t U + \partial_{tt} U(\frac{t(n-1)}{R} + 1) \\
&+ \partial_{zz} U(\frac{t(n-1)}{R} + 1) + \frac{\alpha}{z}\partial_z U(\frac{t(n-1)}{R} + 1) \\
&= \frac{2(n-1)}{R}\partial_t U(t, z),
\end{aligned}$$

and

$$\partial_t V_R(t, z) = \partial_t U(t, z)(\frac{t(n-1)}{R} + 1) + \frac{n-1}{R}U(t, z). \tag{2.8.11}$$

To prove v_R is a subsolution to (2.8.10) in $(B_2^{n+1})^+$, we need to show that

$$\frac{2(n-1)}{R}\partial_t U - \frac{n-1}{\rho}[(\frac{t(n-1)}{R} + 1)\partial_t U + \frac{n-1}{R}U] \geq 0$$

evaluated at $(R-\rho, z)$. Set $t = R - \rho$, the inequality is reduced to

$$[2(R-t) - R - (n-1)t]\partial_t U - (n-1)U \geq 0. \tag{2.8.12}$$

To prove this, an inequality for function U is required as

$$r\frac{\partial_t U(t,z)}{U(t,z)} \geq C > 0, \tag{2.8.13}$$

with $r^2 = t^2 + z^2$. The proof of (2.8.13) is given in Section 2.12.1 in the Appendix.

Then we can show when R is large enough, the inequality (2.8.12) is satisfied.

Next we want to prove that v_R satisfies the free boundary condition. First observe that

$$F(v_R) = \partial B_R^n(Re_n) \cap B_2^n(0),$$

then we want to show

$$v_R(x,z) = aU(x_n,z) + o(|(x,z)|^\beta) \text{ as } (x,z) \to (0,0), \tag{2.8.14}$$

with $a \geq 1$. By the Hölder continuity of U with exponent β, we can see

$$|V_R(t,z) - V_R(t_0,z)| \leq C|t-t_0|^\beta \text{ for } |t-t_0| \leq 1.$$

Thus for $(x,z) \in B_l^{n+1}$, with small $l > 0$,

$$|v_R(x,z) - V_R(x_n,z)| = |V_R(R-\rho,z) - V_R(x_n,z)| \leq C|R-\rho-x_n|^\beta \leq Cl^{2\beta}.$$

Here we use

$$R - \rho - x_n = -\frac{|x'|^2}{R - x_n + \rho}.$$

Then it follows that

$$\begin{aligned}|v_R(x,z)-U(x_n,z)| &\le |v_R(x,z)-V_R(x_n,z)|+|V_R(x_n,z)-U(x_n,z)|\\ &\le Cl^{2\beta}+|U(x_n,z)|(n-1)\frac{|x_n|}{R}\\ &\le Cl^{2\beta}+\tilde{C}l^{\beta+1}\\ &= o(l^\beta)\end{aligned}$$

This gives the desired expansion (2.8.14) with $a=1$.

In the last part, we need to show that

$$\lim_{z\to 0} z^\alpha \partial_z v_R(x,z) \ge \gamma v_R^{\gamma-1}(x,0) \tag{2.8.15}$$

for all $x\in\{v_R(x,0)>0\}\cap B_1^n$. From our definition of v_R, $x\in\{v_R(x,0)>0\}$ means $t=R-\rho>0$. We prove (2.8.15) by showing

$$\begin{aligned}\lim_{z\to 0} z^\alpha\partial_z v_R(x,z) &= \lim_{z\to 0} z^\alpha\partial_z V_R(R-\rho,z)\\ &= (\frac{(n-1)t}{R}+1)\lim_{z\to 0} z^\alpha \partial_z U(R-\rho,z)\\ &= (\frac{(n-1)t}{R}+1)\gamma U^{\gamma-1}(R-\rho,0)\\ &= (\frac{(n-1)t}{R}+1)^{2-\gamma}\gamma v_R^{\gamma-1}(x,0)\\ &\ge \gamma v_R^{\gamma-1}(x,0).\end{aligned}$$

So we complete the proof that v_R is a comparison subsolution to the equation (2.6.1).

Now we show that v_R is strictly monotone increasing in the e_n-direction. Since

$$\partial_{x_n} v_R(x) = -\frac{x_n-R}{\rho}\partial_t V_R(R-\rho,z),$$

so we only need to show $\partial_t V_R(R-\rho, z) > 0$, which follows from (2.8.11) and (2.8.13).

Step 2. In this part we prove the existence of $\tilde{v}_R$ satisfying (2.8.8) and (2.8.9).

First we show there exists unique $\tilde{t}$ such that

$$U(t,z) = V_R(t+\tilde{t}, z) \quad \text{in} \quad (B_1^2)^+ \tag{2.8.16}$$

and

$$|\tilde{t} + \frac{2(n-1)tr}{R}| \leq \frac{\tilde{C}}{R^2} r^3, \tag{2.8.17}$$

with $r^2 = t^2 + z^2$ and universal $\tilde{C}$. Since V_R is strictly increasing in $t-$direction except $\{(t,0), t \leq 0\}$, it suffices to show

$$V_R(t - \frac{2(n-1)tr}{R} - \frac{\tilde{C}}{R^2} r^3) < U(t,z) < V_R(t - \frac{2(n-1)tr}{R} + \frac{\tilde{C}}{R^2} r^3). \tag{2.8.18}$$

To prove this, let

$$\bar{t} = -\frac{2(n-1)tr}{R} - \frac{\tilde{C}}{R^2} r^3$$

and then

$$V_R(t+\bar{t}, z) = V_R(t,z) + \bar{t}\partial_t V_R(t,z) + \frac{1}{2} E|\bar{t}|^2 \tag{2.8.19}$$

with

$$|E| \leq |\partial_{tt} V_R(\tau, z)|, t + \bar{t} < \tau < t.$$

Claim that

$$|\partial_{tt} V_R(\tau, z)| \leq \frac{C'}{r^2} U(t,z). \tag{2.8.20}$$

Following is the proof of (2.8.20).

$$\begin{aligned}\partial_{tt}V_R(\tau,z) &= \frac{n-1}{R}U_t + (\frac{(n-1)\tau}{R}+1)U_{tt} + \frac{n-1}{R}U_t \\ &= 2\frac{n-1}{R}r^{\beta-1}U_t(\frac{\tau}{r},\frac{z}{r}) + (\frac{(n-1)\tau}{R}+1)r^{\beta-2}U_{tt}(\frac{\tau}{r},\frac{z}{r}),\end{aligned}$$

using U is homogeneous of degree β. Since τ is between t and $t+\bar{t}$, so $(\frac{\tau}{r},\frac{z}{r}) \in B^+_{3/2}/B^+_{1/2}$. Here we claim that

$$|\partial_t U(\frac{\tau}{r},\frac{z}{r})| \le K_2 U(\frac{\tau}{r},\frac{z}{r}), \tag{2.8.21}$$

and

$$|\partial_{tt} U(\frac{\tau}{r},\frac{z}{r})| \le K_1 U(\frac{\tau}{r},\frac{z}{r}). \tag{2.8.22}$$

The proofs of these two inequalities are given in Section 2.12.2 and Section 2.12.3 in the Appendix. Then

$$\begin{aligned}|\partial_{tt}V_R(\tau,z)| &\le 2\frac{n-1}{R}r^{\beta-1}K_2U(\frac{\tau}{r},\frac{z}{r}) + (\frac{(n-1)\tau}{R}+1)r^{\beta-2}K_1U(\frac{\tau}{r},\frac{z}{r}) \\ &\le \bar{C}r^{\beta-2}U(\frac{\tau}{r},\frac{z}{r}).\end{aligned}$$

Now what we want to prove is

$$U(\frac{\tau}{r},\frac{z}{r}) \le KU(\frac{t}{r},\frac{z}{r}), \tag{2.8.23}$$

and then we can show

$$|\partial_{tt}V_R(\tau,z)| \le \bar{C}r^{\beta-2}U(\frac{\tau}{r},\frac{z}{r}) \le \bar{C}Kr^{-2}U(t,z).$$

In Section 2.12.4 in the Appendix a proof of (2.8.23) is given, and our claim (2.8.20) is now proved. Using (2.8.19) with the claim (2.8.20), we can prove the lower bound in (2.8.18) if we prove the following

$$U(t,z) > V_R(t,z) + \bar{t}\partial_t V_R(t,z) + \frac{C'}{2r^2}U(t,z)|\bar{t}|^2,$$

and it is equivalent to prove

$$U(t,z) > U(t,z)(\frac{(n-1)t}{R}+1) + \bar{t}((\frac{(n-1)t}{R}+1)U_t(t,z) + \frac{n-1}{R}U(t,z))$$
$$+ \frac{C'}{2r^2}U(t,z)|\bar{t}|^2.$$

Divide both sides by U and then multiply by r, it is equivalent to show

$$\frac{(n-1)t}{R}r + \bar{t}(\frac{(n-1)r}{R} + [\frac{(n-1)t}{R}+1]r\frac{U_t}{U}) + \frac{C'}{2r}|\bar{t}|^2 < 0.$$

Plug in $\bar{t} = -\frac{2(n-1)tr}{R} - \frac{\tilde{C}}{R^2}r^3$, it is equivalent to show

$$\bar{t}[\frac{(n-1)r}{R} - 1/2 + (r\frac{U_t}{U})(\frac{(n-1)t}{R}+1)] + \frac{C'}{2r}|\bar{t}|^2 < \frac{\tilde{C}}{2R^2}r^3.$$

By what we proved in (2.8.13), and for R large enough such that

$$|\bar{t}| \leq Kr^2/R,$$

we can show the above inequality is right for appropriate universal $\tilde{C}$ and R large enough, thus lower bound in (2.8.18) is proved.

To conclude, we use $R-\rho-x_n = -\frac{|x'|^2}{R-x_n+\rho}$ with $\rho = \sqrt{|x'|^2+(x_n-R)^2}$ to write

$$v_R(X-\tilde{v}_R e_n) = V_R(R-\rho(\tilde{v}_R), z) = V_R(x_n - \tilde{v}_R - \frac{|x'|^2}{R-x_n+\tilde{v}_R+\rho(\tilde{v}_R)}, z),$$

with $\rho(\eta) = \sqrt{|x'|^2+(x_n-\eta-R)^2}$. In view of (2.8.16), if there exists $\tilde{v}_R = \tilde{v}_R(X)$ such that

$$-\tilde{v}_R - \frac{|x'|^2}{R-x_n+\tilde{v}_R+\rho(\tilde{v}_R)} = \tilde{t}, \qquad (2.8.24)$$

then

$$U(X) = v_R(X - \tilde{v}_R e_n),$$

and by the strict monotonicity of v_R in e_n direction, $\tilde{v}_R$ must be unique. Thus, the proposition is proved if we show that there exists $\tilde{v}_R$ satisfying (2.8.24) and

$$|\tilde{v}_R(X) - \gamma_R(X)| \leq C\frac{|X|^2}{R^2}.$$

To do so, we define

$$f(\eta) = -\eta - \frac{|x'|^2}{R - x_n + \eta + \rho(\eta)}, -1 \leq \eta \leq 1,$$

and we show that

$$f(\gamma_R(X) + C\frac{|X|^2}{R^2}) \leq \tilde{t} \leq f(\gamma_R(X) - C\frac{|X|^2}{R^2}).$$

Using (2.8.17) we only need to prove

$$f(\gamma_R(X) + C\frac{|X|^2}{R^2}) \leq -\frac{2(n-1)x_n r}{R} - \tilde{C}\frac{r^3}{R^2},$$

and

$$f(\gamma_R(X) - C\frac{|X|^2}{R^2}) \geq -\frac{2(n-1)x_n r}{R} + \tilde{C}\frac{r^3}{R^2}.$$

To prove the first inequality (the second one follows similarly), we define

$$\bar{\eta} = \gamma_R(X) + C\frac{|X|^2}{R^2},$$

and from the definition of f and γ_R, it is equivalent to show

$$\frac{|x'|^2}{2R} - C\frac{|X|^2}{R^2} - \frac{|x'|^2}{R - x_n + \bar{\eta} + \rho(\bar{\eta})} \leq -\tilde{C}\frac{r^3}{R^2}.$$

Since $-1 \leq \bar{\eta} \leq 1$,

$$R - x_n + \bar{\eta} + \rho(\bar{\eta}) \leq 2R + 5$$

and the inequality is reduced to

$$-C\frac{|X|^2}{R^2} + \frac{|x'|^2}{R^2} \leq -\tilde{C}\frac{r^3}{R^2},$$

which is satisfied as long as $C - \tilde{C} \geq 1$. □

Then we can easily obtain the following Corollary.

Corollary 2.8.6. *There exist universal constants δ, c_0, C_0, C_1 such that*

$$v_R(X + \frac{c_0}{R}e_n) \leq (1 + \frac{C_0}{R})U(X) \quad in \quad \overline{(B_1^{n+1})^+}/B_{1/4} \tag{2.8.25}$$

with strict inequality on $F(v_R(X + \frac{c_0}{R}e_n)) \cap (\overline{(B_1^{n+1})^+}/B_{1/4})$, and

$$v_R(X + \frac{c_0}{R}e_n) \geq U(X + \frac{c_0}{2R}e_n) \quad in \quad (B_\delta^{n+1})^+, \tag{2.8.26}$$

$$v_R(X - \frac{C_1}{R}e_n) \leq U(X) \quad in \quad \overline{(B_1^{n+1})^+}. \tag{2.8.27}$$

Now we prove Lemma 2.8.3. We prove the first statement, and the second one follows similarly.

Proof. In view of (2.8.1),

$$g(\bar{X}) - U(\bar{X}) \geq U(\bar{X} + \epsilon e_n) - U(\bar{X}) = \partial_t U(\bar{X} + \lambda e_n)\epsilon \geq c\epsilon$$

for $\lambda \in (0, \epsilon)$. From Lemma 2.8.4, we get

$$g(X) \geq (1 + c'\epsilon)U(X) \quad \text{in} \quad (B_{1/4}^{n+1})^+. \tag{2.8.28}$$

Now let $R = \frac{C_0}{c'\epsilon}$, C_0 is the constant in Corollary 2.8.6. Then for ϵ small enough, v_R is a subsolution to (2.6.1) in $(B_2^{n+1})^+$ which is monotone increasing in the e_n-direction and it also satisfies inequalities in Corollary 2.8.6. We now apply the comparison principle stated in Corollary 2.6.7. Let

$$v_R^t(X) = v_R(X + te_n)$$

and according to (2.8.27),

$$v_R^{t_0} \leq U \leq g \quad \text{in} \quad (B_{1/4}^{n+1})^+,$$

with $t_0 = -C_1/R$. Moreover, from (2.8.25) to (2.8.28), we get that for our choice of R,

$$v_R^{t_1} \leq (1 + c'\epsilon)U \leq g \quad \text{in} \quad \partial B_{1/4}^{n+1} \cap \{y \geq 0\},$$

with $t_1 = c_0/R$, with strict inequality on $F(v_R^{t_1}) \cap \partial B_{1/4}^{n+1} \cap \{y \geq 0\}$. In particular,

$$g > 0 \quad \text{on} \quad \mathcal{G}(v_R^{t_1}) \cap (B_{1/4}^{n+1})^+.$$

Thus we can apply the comparison principle to prove

$$v_R^{t_1} \leq g \quad \text{in} \quad (B_{1/4}^{n+1})^+.$$

From (2.8.26) we obtain

$$U(X + \frac{c_1}{R}e_n) \leq v_R^{t_1}(X) \leq g(X) \quad \text{in} \quad (B_\delta^{n+1})^+,$$

which is desired in (2.8.2) with $\tau = \frac{c_1 c'}{C_0}$. $\square$

2.9 Improvement of flatness

In this section we show the proof of the improvement of flatness property for solutions to (2.6.1).

Theorem 2.9.1 (Improvement of flatness)**.** *There exists $\bar{\epsilon} > 0$ and $\rho > 0$ universal constants such that for all $0 < \epsilon < \bar{\epsilon}$, if g solves (2.6.1) with $0 \in F(g)$ and it satisfies*

$$U(X - \epsilon e_n) \leq g(X) \leq U(X + \epsilon e_n) \quad in \quad (B_1^{n+1})^+, \tag{2.9.1}$$

then

$$U(x \cdot \nu - \epsilon\rho/2, z) \leq g(X) \leq U(x \cdot \nu + \epsilon\rho/2, z) \quad in \quad (B_\rho^{n+1})^+, \tag{2.9.2}$$

for some direction $\nu \in \mathbb{R}^n$, $|\nu| = 1$.

The proof of Theorem 2.9.1 is divided into the next four lemmas.

The following lemma is the same as in Lemma 7.2 in [16] and its proof remains unchanged since it only depend on elementary properties related to the definition of $\tilde{g}_\epsilon$, and does not depend on the equations satisfied by g.

Lemma 2.9.2. *Let g be a solution to (2.6.1) with $0 \in F(g)$ and g satisfies (2.9.1). Assume that*

$$a_0 \cdot x' - \rho/4 \leq \tilde{g}_\epsilon(X) \leq a_0 \cdot x' + \rho/4 \quad in \quad (B_{2\rho}^{n+1})^+, \tag{2.9.3}$$

for some $a_0 \in \mathbb{R}^{n-1}$. Then if $\epsilon \leq \bar{\epsilon}(a_0, \rho)$, g satisfies (2.9.2) in $(B_\rho^{n+1})^+$.

The next lemma follows immediately from Corollary 2.8.2.

Lemma 2.9.3. *Let $\epsilon_k \to 0$ and let g_k be a sequence of solutions to* (2.6.1) *with $0 \in F(g_k)$ satisfying*

$$U(X - \epsilon_k e_n) \le g_k(X) \le U(X + \epsilon_k e_n) \quad in \quad (B_1^{n+1})^+. \tag{2.9.4}$$

Let $\tilde{g}_k$ denote the e_k-domain variation of g_k. Then the sequence of sets

$$A_k := \{(X, \tilde{g}_k(X)) : X \in B_{1-\epsilon_k}^{n+1}\},$$

has a subsequence that converges uniformly in Hausdorff distance in $(B_{1/2}^{n+1})^+$ to the graph

$$A_\infty := \{(X, \tilde{g}_\infty(X)) : X \in (B_{1/2}^{n+1})^+\},$$

where $\tilde{g}_\infty$ is Hölder continuous.

Lemma 2.9.4. *The limiting function satisfies $\tilde{g}_\infty \in C_{loc}^{1,1}(B_{1/2}^{n+1})^+$.*

Proof. We fix a point $Y \in (B_{1/2}^{n+1})^+$, and let δ be the distance from Y to $L = \{x_n = 0, y = 0\}$. It suffices to show that the function $\tilde{g}_{\epsilon_k}$ are uniformly $C^{1,1}$ in $B_{\delta/8}^{n+1}(Y) \cap \{y > 0\}$. Since $g_k - U$ solves

$$div(y^\alpha \nabla (g_k - U)) = 0 \;\text{ in }\; B_{\delta/2}^{n+1}(Y) \cap \{y > 0\},$$

we can see

$$\|g_k - U\|_{C^{1,1}(B_{\delta/4}^{n+1}(Y)\cap\{y>0\})} \le C\|g_k - U\|_{L^\infty(B_{\delta/2}^{n+1}(Y)\cap\{y>0\})} \le C\epsilon,$$

and by implicit function theorem it follows as

$$\|\tilde{g}_{\epsilon_k}\|_{C^{1,1}(B_{\delta/8}^{n+1}(Y)\cap\{y>0\})} \le C,$$

with constant C depending on Y and δ. $\square$

Lemma 2.9.5. *The function* $\tilde{g}_\infty$ *solves the linearized problem* (2.7.3) *in* $(B_{1/2}^{n+1})^+$.

Proof. We start by showing that in the sense of viscosity, $U_n\tilde{g}_\infty$ satisfies

$$div(z^\alpha \nabla(U_n\tilde{g}_\infty)) = 0 \text{ in } (B_{1/2}^{n+1})^+.$$

Let $\tilde{\phi}$ be a C^2 function touching $\tilde{g}_\infty$ by below at $X_0 = (x_0, z_0) \in (B_{1/2}^{n+1})^+$, and we want to show that

$$\Delta(U_n\tilde{\phi})(X_0) + \alpha \frac{\partial_z(U_n\tilde{\phi})(X_0)}{z_0} \leq 0, \tag{2.9.5}$$

and here the Laplace operator Δ is in $\mathbb{R}^{n+1}$.

By Lemma 2.9.3, the sequence A_k converges uniformly to A_∞, thus there exists a sequence of constants $c_k \to 0$ and a sequence of points $X_k \to X_0$ such that $\tilde{\phi}_k := \tilde{\phi} + c_k$ touches $\tilde{g}_k$ by below at X_k for k large enough.

Define ϕ_k by below

$$\phi_k(X - \epsilon_k\tilde{\phi}_k(X)e_n) = U(X). \tag{2.9.6}$$

Then according to (2.7.2), ϕ_k touches g_k by below at $Y_k = X_k - \epsilon_k\tilde{\phi}_k(X_k)e_n$, for k large enough. Thus, since g_k solves

$$div(z^\alpha \nabla g_k) = 0 \text{ in } (B_1^{n+1})^+,$$

it follows that

$$\Delta(\phi_k)(Y_k) + \alpha \frac{\partial_{n+1}(\phi_k)(Y_k)}{z_k} \leq 0. \tag{2.9.7}$$

Here let ∂_{n+1} denote the (n+1)-th derivative (same as ∂_z), and z_k is the (n+1)-th coordinate of Y_k. Let $U_n = \partial_n U$ denote the n-th derivative of U, and

$U_z = \partial_z U = \partial_{n+1} U$. Now we compute $\Delta(\phi_k)(Y_k)$ and $\partial_{n+1}(\phi_k)(Y_k)$. Since $\tilde{\phi}$ is smooth, for any Y in a neighborhood of Y_k, there exists a unique $X = X(Y)$ such that

$$Y = X - \epsilon_k \tilde{\phi}_k(X) e_n. \tag{2.9.8}$$

Thus (2.9.6) reads as

$$\phi_k(Y) = U(X(Y)),$$

with $Y_i = X_i$ if $i \neq n$ and when $j \neq n$,

$$\frac{\partial X_j}{\partial Y_i} = \delta_{ij}.$$

Then

$$D_X Y = I - \epsilon_k D_X(\tilde{\phi}_k(X) e_n), \tag{2.9.9}$$

and

$$D_Y X = I + \epsilon_k D_X(\tilde{\phi} e_n) + O(\epsilon_k^2), \tag{2.9.10}$$

since

$$\tilde{\phi}_k = \tilde{\phi} + c_k.$$

It follows that

$$\frac{\partial X_n}{\partial Y_j} = \delta_{jn} + \epsilon_k \partial_j \tilde{\phi}(X) + O(\epsilon_k^2). \tag{2.9.11}$$

Then we can compute

$$\begin{aligned} \Delta\phi_k(Y) = {} & U_n(X)\Delta X_n(Y) \\ & + \sum_{j\neq n}(U_{jj}(X) + 2U_{jn}\frac{\partial X_n}{\partial Y_j}) \\ & + U_{nn}(X)|\nabla X_n|^2(Y). \end{aligned} \tag{2.9.12}$$

By (2.9.11), we can compute

$$|\nabla X_n|^2(Y) = 1 + 2\epsilon_k \partial_n \tilde{\phi}(X) + O(\epsilon_k^2),$$

and

$$\begin{aligned} \frac{\partial^2 X_n}{\partial Y_j^2} &= \epsilon_k \sum_i \partial_{ji}\tilde{\phi}\frac{\partial X_i}{\partial Y_j} + O(\epsilon_k^2) \\ &= \epsilon_k \sum_{i \neq n} \partial_{ji}\tilde{\phi}\delta_{ij} + \epsilon_k \partial_{jn}\tilde{\phi}\frac{\partial X_n}{\partial Y_j} + O(\epsilon_k^2). \end{aligned} \tag{2.9.13}$$

Then

$$\Delta X_n = \epsilon_k \Delta\tilde{\phi} + O(\epsilon_k^2). \tag{2.9.14}$$

Using (2.9.14) and (2.9.13) in (2.9.12), we can get

$$\Delta\phi_k(Y) = \Delta U(X) + \epsilon_k U_n \Delta\tilde{\phi} + 2\epsilon_k \nabla\tilde{\phi}\cdot\nabla U_n + O(\epsilon_k^2)(U_n n + 2\sum_{j\neq n} U_{jn}). \tag{2.9.15}$$

We can also compute that

$$\begin{aligned} (\phi_k)_{n+1}(Y) &= U_n(X)\frac{\partial X_n}{\partial Y_{n+1}} + U_z(X)\frac{\partial X_{n+1}}{\partial Y_{n+1}} \\ &= U_n(X)(\epsilon_k \partial_{n+1}\tilde{\phi}(X) + O(\epsilon_k^2)) + U_z(X). \end{aligned} \tag{2.9.16}$$

Plug in (2.9.15), and $\Delta U(X_k) + \frac{\alpha}{z} U_z(X_k) = 0$ to (2.9.7), we can compute that

$$\epsilon_k(U_n\Delta\tilde{\phi} + 2\nabla\tilde{\phi}\nabla U_n + \Delta U_n\tilde{\phi} + \frac{\alpha}{z_k}U_n\partial_{n+1}\tilde{\phi} + \frac{\alpha}{z_k}(U_n)_z\tilde{\phi}) + O(\epsilon_k^2) \leq 0, \tag{2.9.17}$$

which means

$$\Delta(U_n\tilde{\phi})(X_k) + \frac{\alpha}{z_k}\partial_z(U_n\tilde{\phi})(X_k) + O(\epsilon_k) \leq 0.$$

Then the desired inequality (2.9.5) follows as $k \to \infty$.

The next step is to show that $\tilde{g}_\infty$ solves

$$\lim_{z\to 0} z^\alpha \partial_z \tilde{g}_\infty = 0 \quad \text{on} \quad \{x_n > 0\} \cap B_1^n. \tag{2.9.18}$$

Since ϕ_k touches g_k by below at Y_k and g_k solves (2.9.18), we have

$$\lim_{z\to 0} z^\alpha \partial_z \phi_k(Y_k) \geq \gamma \phi_k^{\gamma-1}(Y_k),$$

and by the computations in the previous part,

$$\partial_z \phi_k(Y_k) = U_n(X)(\epsilon_k \partial_{n+1} \tilde{\phi}(X_k) + O(\epsilon_k^2)) + U_z(X_k),$$

therefore,

$$\begin{aligned} \gamma \phi_k^{\gamma-1}(Y_k) &\leq \partial_z \phi_k(Y_k) \\ &= z^\alpha U_n \partial_{n+1} \tilde{\phi}(X_k) \epsilon_k + O(\epsilon_k^2) U_n(X_k) + z^\alpha \partial_z U(X_k). \end{aligned} \tag{2.9.19}$$

Since

$$\phi_k(Y_k) = U(X_k)$$

as defined and U satisfies

$$\lim_{z\to 0} z^\alpha \partial_z U = \gamma U^{\gamma-1},$$

we can show

$$\epsilon_k U_n z^\alpha \partial_{n+1} \tilde{\phi}(X_k) + O(\epsilon_k^2) U_n(X_k) \geq 0$$

and thus

$$z^\alpha \partial_{n+1} \tilde{\phi}(X_k) \geq 0.$$

Here we use U_n is strictly monotonuous increasing in the e_n-direction in $B_1^{n+1} \cap \{y \geq 0\} \setminus P$. Since $\tilde{\phi}_k = \tilde{\phi} + c_k$ touches $\tilde{g}_k$ by below, letting $k \to \infty$, we can prove that $\tilde{g}_\infty$ solves (2.9.18) on $\{x_n > 0\} \cap B_1^n$.

Then we want to show that $\tilde{g}_\infty$ solves

$$|\nabla_r \tilde{g}_\infty|(X_0) = 0, X_0 = (x_0', 0, 0) \in B_{1/2}^n \cap L. \tag{2.9.20}$$

Assume by contradiction, there exists ψ touching by below at X_0 and

$$\psi(X) = \psi(X_0) + a(X_0)(x' - x_0') + b(X_0)r + O(|x' - x_0'|^2 + r^{1+l})$$

for some $l > 0$ and $b(X_0) > 0$. Then there exists $\theta, \delta, \bar{r}$ and $Y' = (y_0', 0, 0) \in B_2$ depending on ψ such that

$$q(X) = \psi(X_0) - \frac{\theta}{2}|x' - y_0'|^2 + 2\theta(n-1)x_n r$$

which is a second order polynomial touches ψ by below at X_0, in a neighborhood $N_{\bar{r}} = \{|x' - x_0'| \leq \bar{r}, r \leq \bar{r}\}$ of X_0. Also $\psi - q \geq \delta > 0$ on $N_{\bar{r}}/N_{\bar{r}/2}$. Then we can see

$$\tilde{g}_\infty - q \geq \delta > 0 \quad \text{on} \quad N_{\bar{r}} \setminus N_{\bar{r}/2},$$

and

$$\tilde{g}_\infty(X_0) - q(X_0) = 0.$$

In particular,

$$|\tilde{g}_\infty(X_k) - q(X_k)| \to 0, X_k \in N_{\bar{r}} \setminus \{x_n \leq 0, z = 0\}, X_k \to X_0.$$

Now choose $R_k = \frac{1}{\theta \epsilon_k}$ and and define

$$w_k(X) = v_{R_k}(X_Y' + \epsilon_k \psi(X_0) e_n), Y = (y_0', 0, 0),$$

with v_R defined in (2.8.7). Then the ϵ_k domain variation of w_k can be defined by

$$w_k(X - \epsilon_k \tilde{w}_k(X) e_n) = U(X),$$

and since U is invariant in x'-direction, this is equivalent to

$$v_{R_k}(X - Y' + \epsilon_k \psi(X_0) e_n - \epsilon_k \tilde{w}_k(X) e_n) = U(X - Y').$$

Proposition 2.8.5 leads to

$$\tilde{v}_{R_k}(X - Y') = \epsilon_k(\tilde{w}_k(X) - \psi(X_0)).$$

Then we can conclude from (2.8.9) that

$$\tilde{w}_k(X) = q(X) + \theta^2 \epsilon_k O(|X - Y'|^2),$$

and hence

$$|\tilde{w}_k - q| \leq C\epsilon_k \quad \text{on} \quad N_{\bar{r}} \setminus \{x_n \leq 0, z = 0\}$$

Thus from the uniform convergence of A_k to A_∞, we get for k large enough,

$$\tilde{g}_k - \tilde{w}_k \geq \delta/2 \quad \text{on} \quad (N_{\bar{r}} \setminus N_{\bar{r}/2}) \setminus \{x_n \leq 0, z = 0\}. \tag{2.9.21}$$

Similarly we can get

$$\tilde{g}_k(X_k) - \tilde{w}_k(X_k) \leq \delta/4,$$

for some sequence $X_k \in N_{\bar{r}} \setminus \{x_n \leq 0, z = 0\}$, and $X_k \to X_0$.

However from Lemma 2.7.1 and (2.9.21), we can see

$$\tilde{g}_k - \tilde{w}_k \geq \delta/2 \quad \text{on} \quad N_{\bar{r}}/\{x_n \leq 0, z = 0\}$$

which leads to a contradiction.

We complete the proof of Lemma 2.9.5 that $\tilde{g}_\infty$ solves the linearized problem (2.7.3) in $(B^{n+1}_{1/2})^+$. $\square$

We need regularity of the solutions to the linearized problem (2.7.3) (in Section 2.10) to finish the proof of Theorem 2.9.1 in Section 2.11, and then the proof of Theorem 2.6.8 follows in that section.

2.10 The regularity of the solutions of the linearized problem

In this section, our aim is to prove the regularity results for w solving the following linearized equation in the case γ is small enough.

$$\begin{cases} div(y^\alpha \nabla((U_\gamma)_n w)) = 0 \ \text{ in } \ (B_1^{n+1})^+, \\ |\nabla_r w|(X_0) = 0 \ \text{ on } \ B_1^n \cap L, \\ \lim_{y\to 0^+} y^\alpha \partial_y w(x,y) = 0 \ \text{ on } \ B_1^n \cap \{x_n > 0\}. \end{cases} \tag{2.10.1}$$

Here we let the function U_γ denote the extension of $(x_n)_+^\beta$ to upper half space $(\mathbb{R}^{n+1})^+$, and the exponent $\beta = \frac{2s}{2-\gamma}$ depends on γ.

The following theorem states the regularity results for the solutions of the linearized problem when γ is close to 0.

Theorem 2.10.1. *There exists $\gamma_0 > 0$, such that for all $0 < \gamma < \gamma_0$, the following regularity result holds.*

Given a boundary data $\bar{h} \in C((\partial B_1^{n+1})^+)$, $|\bar{h}| \leq 1$, then there exists a unique classical solution h to (2.10.1) *such that $h \in C(\overline{(B_1^{n+1})^+})$, $h = \bar{h}$ on $(\partial B_1^{n+1})^+$, and it satisfies*

$$|h(X) - h(X_0) - a' \cdot (x' - x_0')| \leq C(|x' - x_0'|^2 + r^{1+\theta}), X_0 \in B_{1/2}^{n+1} \cap L, \quad (2.10.2)$$

for universal constants C, θ and a vector $a' \in \mathbb{R}^{n-1}$ depending on X_0.

A corollary of the theorem above is what we need in the proof of Theorem 2.6.8.

Corollary 2.10.2. *There exists a universal constant C such that if w is a viscosity solution to* (2.10.1), *with*

$$-1 \leq w(X) \leq 1 \quad in \quad (B_1^{n+1})^+,$$

then

$$a_0 \cdot x' - C|X|^{1+\theta} \leq w(X) - w(0) \leq a_0 \cdot x' + C|X|^{1+\theta},$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

From Corollary 2.10.2, there exists $\rho > 0$, if w is a viscosity solution to (2.10.1), with $w(0) = 0$ and

$$-1 \leq w(X) \leq 1 \quad \text{in} \quad (B_1^{n+1})^+,$$

then

$$a_0 \cdot x' - \frac{1}{8}\rho \leq w(X) \leq a_0 \cdot x' + \frac{1}{8}\rho, \quad \text{in} \quad (B_{2\rho}^{n+1})+ \quad (2.10.3)$$

for some vector $a_0 \in \mathbb{R}^{n-1}$.

The proof of Theorem 2.10.1 is based on the method of compactness. In paper [13] section 6, Theorem 6.1 states the same results for the linearized problem of the limiting case $\gamma = 0$. In the $\gamma = 0$ case, w solves

$$\begin{cases} div(y^\alpha \nabla((U_0)_n w)) = 0 \ \text{ in } \ (B_1^{n+1})^+, \\ |\nabla_r w|(X_0) = 0 \ \text{ on } \ B_1^n \cap L, \\ \lim_{y\to 0^+} y^\alpha \partial_y((U_0)_n w(x,y)) = 0 \ \text{ on } \ B_1^n \cap \{x_n > 0\}. \end{cases} \tag{2.10.4}$$

with $U_0(X) = U_0(x_n, y) = (r^{1/2}\cos(\theta/2))^{2s}$, $r^2 = x_n^2 + y^2$. The regularity is stated same in Theorem 2.10.1. Our aim is to use the method of compactness to prove Theorem 2.10.1 for $0 < \gamma < \gamma_0$ small enough.

Proof. If not, then there exists a sequence $\gamma_k \to 0$ such that given boundary data $\bar{h}$ and $|\bar{h}| \le 1$, w_k solves (2.10.1) for $\gamma = \gamma_k$ with boundary data $\bar{h}$, and for any $a' \in \mathbb{R}^{n-1}$, and for any $C > 0, \theta > 0$, there exists $X_k, \tilde{X}_k \in B_{1/2}^n \cap L$, such that

$$|w_k(\tilde{X}_k) - w_k(X_k) - a'(x_k' - \tilde{x}_k')| > C(|x_k' - \tilde{x}_k'|^2 + r^{1+\theta}).$$

Consider the limits of the subsequences (denoted by γ_k, X_k, and $\tilde{X}_k$ as well), such that $\tilde{X}_k \to \tilde{X}_0$, $X_k \to X_0$, and $w_k \to w_0$. Then $w_0 = \bar{h}$ on $(\partial B_1^{n+1})^+$ and for any $a' \in \mathbb{R}^{n-1}$, and for any $C > 0, \theta > 0$,

$$|w_0(\tilde{X}_0) - w_0(X_0) - a'(x_0' - \tilde{x}_0')| > C(|x_0' - \tilde{x}_0'|^2 + r^{1+\theta}).$$

Now we want to prove the limit w_0 solves (2.10.4), and then it leads to a contradiction.

Let

$$J_0(w) = \int_{(B_1^{n+1})^+} y^\alpha (U_0)_n^2 |\nabla w|^2 dX.$$

Then as proved in section 6 in [13], the minimizer of the energy $J_0(w)$ solves

$$div(y^\alpha (U_0)_n^2 \nabla w) = 0 \ \text{ in } \ (B_1^{n+1})^+, \tag{2.10.5}$$

and (2.10.5) is equivalent to

$$div(y^\alpha \nabla((U_0)_n w)) = 0 \ \text{ in } \ (B_1^{n+1})^+. \tag{2.10.6}$$

Moreover, it is proved in section 6 in [13] that if w solves (2.10.6), and

$$\lim_{r \to 0} w_r(x', x_n, y) = b(x'), \quad \text{on} \quad L \cap B_1^n,$$

then w is a minimizer of $J_0(w)$ if and only if $b \equiv 0$.

Therefore, let w_k be the solution to (2.10.1) for $\gamma = \gamma_k$. Then w_k is a minimizer of $J_{\gamma_k}(w) = \int_{(B_1^{n+1})^+} y^\alpha (U_{\gamma_k})_n^2 |\nabla w|^2 dX$, and w_k satisfies

$$\lim_{y \to 0^+} y^\alpha \partial_y ((U_{\gamma_k})_n w_k(x,y)) = w_k(x,0) \lim_{y \to 0^+} y^\alpha \partial_y (U_{\gamma_k})_n.$$

This equality is derived from $\lim_{y \to 0^+} y^\alpha \partial_y w_k(x,y) = 0$.

We have $\lim_{\gamma_k \to 0} U_{\gamma_k} = U_0$ in $C((B_1^{n+1})^+)$, and thus if w_k is a minimizer of $J_{\gamma_k}(w)$, then $w_0 = \lim_{\gamma_k \to 0} w_k$ is a minimizer of J_0. The limit w_0 also satisfies (2.10.6), therefore,

$$(w_0)_r(x', x_n, y) = 0 \ \text{ on } \ L \cap B_1^n. \tag{2.10.7}$$

Moreover, since $w_k \to w_0$,

$$\lim_{y\to 0^+} y^\alpha \partial_y((U_{\gamma_k})_n w_k(x,y)) = w_k(x,0) \lim_{y\to 0^+} y^\alpha (U_{\gamma_k})_n,$$

and

$$\lim_{y\to 0^+} y^\alpha \partial_y(\lim_{\gamma_k\to 0} U_{\gamma_k})_n = \lim_{y\to 0^+} y^\alpha \partial_y (U_0)_n = 0,$$

we can prove

$$\lim_{y\to 0^+} y^\alpha \partial_y((U_0)_n w_0(x,y)) = 0.$$

Therefore, the limit w_0 solves (2.10.4), which leads to a contradiction. Now Theorem 2.10.1 is proved and Corollary 2.10.2 follows. □

2.11 Proof of Theorem 2.6.8

In this section, we apply the regularity results of the linearized problem (2.7.3) to prove Theorem 2.9.1. Then the proof of Theorem 2.6.8 simply follows by Theorem 2.9.1 and Lemma 2.6.9.

Following is the proof of Theorem 2.9.1.

Proof. Let ρ be the universal constant in (2.10.3), and assume by contradiction that there exists $\epsilon_k \to 0$ and a sequence of solutions g_k to (2.6.1) such that g_k satisfies

$$U(X - \epsilon_k e_n) \le g_k(X) \le U(X + \epsilon_k e_n) \quad \text{in} \quad (B_1^{n+1})^+, \tag{2.11.1}$$

but it does not satisfy the conclusion of Theorem 2.9.1.

Let $\tilde{g}_k$ denote the ϵ_k-domain variation of g_k. Then by Lemma 2.9.3 the sequence of sets

$$A_k := \{(X, \tilde{g}_k(X)) : X \in (B^{n+1}_{1-\epsilon_k})^+\},$$

converges uniformly to

$$A_\infty = \{(X, \tilde{g}_\infty(X)) : X \in (B^{n+1}_{1/2})^+\},$$

where $\tilde{g}_\infty$ is a Hölder continuous function. By Lemma 2.9.5, the function $\tilde{g}_\infty$ solves the linearized equation (2.7.3), and hence by Corollary 2.10.2,

$$a_0 \cdot x' - \rho/8 \leq \tilde{g}_\infty \leq a_0 \cdot x' + \rho/8 \ \text{ in } \ (B^{n+1}_{2\rho})^+,$$

with $a_0 \in \mathbb{R}^{n-1}$. From the uniform convergence (up to extracting a subsequence) of A_k to A_∞, we get that for all k large enough,

$$a_0 \cdot x' - \rho/4 \leq \tilde{g}_k \leq a_0 \cdot x' + \rho/4 \ \text{ in } \ (B^{n+1}_{2\rho})^+.$$

Then by Lemma 2.9.2, g_k satisfies (2.9.2) when k is large enough, which leads to a contradiction. $\square$

2.12 Appendix

Let $U(t,z) = r^\beta g(\theta) \geq 0$, $r = \sqrt{t^2+z^2}$, $t = r\cos\theta$ and $z = r\sin\theta$, with $\theta \in [0,\pi]$. Since $div(z^\alpha \nabla U) = 0$, and $\lim_{z\to 0} z^\alpha \partial_z U(t,z) = \gamma U^{\gamma-1}(t,0)$, so $g(\theta)$ solves the ODE

$$g''(\theta) + \alpha \cot\theta g'(\theta) + \beta(\alpha+\beta)g(\theta) = 0, \tag{2.12.1}$$

with $g(\pi) = 0$, $g(0) = 1$, and $g(\theta) = 1 + \gamma(sin\theta)^{2s} + o((sin\theta)^{2s})$ as $\theta \to 0$.

2.12.1 Proof of (2.8.13)

In the first part, we prove the following inequality:

$$r\frac{\partial_t U(t,z)}{U(t,z)} \geq C > 0.$$

We can compute that

$$\frac{U_t}{U} = \frac{1}{r}(\beta\cos\theta - \frac{g'(\theta)\sin\theta}{g(\theta)}) =: \frac{1}{r}f(\theta).$$

We define

$$f(\theta) = \beta\cos\theta - \frac{g'(\theta)\sin\theta}{g(\theta)}, \tag{2.12.2}$$

and then

$$f'(\theta) = \frac{1}{\sin\theta}[(f(\theta) - (\beta - s)\cos\theta)^2 + (\beta - s)^2\sin^2\theta - s^2]. \tag{2.12.3}$$

We can get $f(0) = \beta$ since

$$\lim_{\theta\to 0}\frac{g'(\theta)\sin\theta}{g(\theta)} = \lim_{\theta\to 0}\frac{g(\theta) - g(0)}{g(0) + \gamma(\sin\theta)^{2s}} = 0, \tag{2.12.4}$$

and $f(\pi) = 2s - \beta > 0$ since

$$\begin{aligned}\lim_{\theta\to\pi}\frac{g'(\theta)\sin\theta}{g(\theta)} &= \lim_{\theta\to\pi} g'(\theta)(\sin\theta)^{\alpha}\frac{(\sin\theta)^{2s}}{g(\theta)} \\ &= \lim_{\theta\to\pi} g'(\theta)(\sin\theta)^{\alpha}\lim_{\theta\to\pi}\frac{2s(\sin\theta)^{2s-1}\cos\theta}{g'(\theta)} \\ &= -2s.\end{aligned} \tag{2.12.5}$$

In addition, $g'(\theta)(\sin\theta)^\alpha$ is bounded and proof is given in (2.8.5). Also, we can get $f'(0) = 0$ and $f'(\pi) = 0$ by

$$\lim_{\theta\to 0} f'(\theta) = \lim_{\theta\to 0} \frac{2ff' - 2(\beta - s)\cos\theta f' + 2(\beta - s)\sin\theta f}{\cos\theta} = \lim_{\theta\to 0} 2sf'(\theta),$$

and similarly

$$\lim_{\theta\to\pi} f'(\theta) = \lim_{\theta\to 0} -2sf'(\theta).$$

Now we want to prove that $f(\theta) \geq C > 0$ for $\theta \in [0, \pi]$. If not, then with the information of f and f' at the end points, there exists at least one $\theta_0 \in (0, \pi)$ such that

$$\begin{cases} f'(\theta_0) = 0, \\ f(\theta_0) \leq 0 \\ f''(\theta_0) > 0. \end{cases}$$

Since $f'(\theta_0) = 0$,

$$f(\theta_0)^2 - 2(\beta - s)\cos\theta_0 f(\theta_0) + (\beta - s)^2 - s^2 = 0,$$

and thus

$$f(\theta_0) = (\beta - s)\cos\theta_0 \pm \sqrt{s^2 - (\beta - s)^2\sin^2\theta_0}.$$

If it is the plus sign, then

$$f(\theta_0) > (\beta - s)\cos\theta_0 + (\beta - s)|\cos\theta_0| \geq 0$$

which is not right. Thus

$$f(\theta_0) = (\beta - s)\cos\theta_0 - \sqrt{s^2 - (\beta - s)^2\sin^2\theta_0}.$$

Then we can compute $f''(\theta)$ at θ_0, that

$$f''(\theta) = \frac{(2ff' - 2(\beta - s)\cos\theta f' + 2(\beta - s)\sin\theta f)\sin\theta - (f'\sin\theta)\cos\theta}{\sin^2\theta}.$$

When $\theta = \theta_0$,

$$0 < \sin^2\theta f''(\theta_0) = 2(\beta - s)\sin^2\theta_0 f(\theta_0) < 0,$$

which leads to a contradiction.

2.12.2 Proof of (2.8.21)

In this section we prove

$$|U_t(\frac{\tau}{r}, \frac{z}{r})| \leq K_2 U(\frac{\tau}{r}, \frac{z}{r}).$$

with $(\frac{\tau}{r}, \frac{z}{r}) \in B^+_{3/2}/B^+_{1/2}$. Let $\theta = arctan(\frac{z}{\tau}) \in [0, \pi]$. Since U is homogeneous of degree β, we can see

$$\frac{U_t(\frac{\tau}{r}, \frac{z}{r})}{U(\frac{\tau}{r}, \frac{z}{r})} = \frac{r}{\sqrt{\tau^2 + z^2}}(\beta\cos\theta - \frac{g'(\theta)\sin\theta}{g(\theta)}) \leq 2f(\theta)$$

with

$$f(\theta) = \beta\cos\theta - \frac{g'(\theta)\sin\theta}{g(\theta)},$$

which is the same definition as in (2.12.2). As computed in the previous section, $f(0) = \beta$, $f(\pi) = 2s - \beta < \beta$, $f(\theta) \geq C > 0$, and

$$f'(\theta) = \frac{1}{\sin\theta}[(f(\theta) - (\beta - s)\cos\theta)^2 + (\beta - s)^2\sin^2\theta - s^2].$$

If there exists θ_0 such that $f(\theta_0) = +\infty$, then $f' = +\infty$ and cannot be negative infinity at such θ_0, or it leads to a contradiction of $f(0) = \beta$, $f(\pi) = 2s - \beta < \beta$ and $\theta \in [0, \pi]$ which is a bounded interval. Therefore, there must exists an upper bound for $f(\theta)$ and then we can prove

$$|\frac{U_t(\frac{\tau}{r}, \frac{z}{r})}{U(\frac{\tau}{r}, \frac{z}{r})}| \leq K_2.$$

2.12.3 Proof of (2.8.22)

First we prove

$$|\frac{U_{tt}(t,z)}{U(t,z)}| \le \frac{C(s,\gamma)}{r^2}. \tag{2.12.6}$$

Write $U(t,z) = r^\beta g(\theta)$, where $t = r\cos\theta$, $z = r\sin\theta$ and $r = \sqrt{t^2+z^2}$. Then

$$U_t = r^{\beta-2}(\beta g(\theta)t - g'(\theta)z),$$

and

$$U_{tt} = r^{\beta-4}(((\beta^2-\beta)t^2+\beta z^2)g(\theta) + (2-2\beta)tzg'(\theta) + z^2 g''(\theta)).$$

Then

$$r^2\frac{U_{tt}}{U} = (\beta^2-\beta)\cos^2\theta + \beta\sin^2\theta + \frac{g'(\theta)}{g(\theta)}(2-2\beta)\sin\theta\cos\theta + \frac{g''(\theta)}{g(\theta)}\sin^2\theta =: F(\theta).$$

Since $div(z^\alpha \nabla U) = 0$, so $g(\theta)$ solves

$$g''(\theta) + \alpha\cot\theta g'(\theta) + \beta(\alpha+\beta)g(\theta) = 0.$$

Then we can replace $g''(\theta)$ in $F(\theta)$ and compute

$$\begin{aligned} F(\theta) &= (\beta^2-\beta)\cos^2\theta + \beta\sin^2\theta + \frac{g'(\theta)}{g(\theta)}(2-2\beta)\sin\theta\cos\theta \\ &\quad - \frac{\alpha\cot\theta g'(\theta) + \beta(\alpha+\beta)g(\theta)}{g(\theta)}\sin^2\theta \\ &= (\beta^2-\beta)\cos^2\theta + \beta(1-\alpha-\beta)\sin^2\theta + (2-2\beta-\alpha)\sin\theta\cos\theta\frac{g'(\theta)}{g(\theta)}. \end{aligned} \tag{2.12.7}$$

First,

$$F(0) = \beta^2 - \beta + \lim_{\theta\to 0}\frac{g'(\theta)}{g(\theta)}\sin\theta(2-\alpha-2\beta) = \beta^2-\beta$$

since $\lim_{\theta\to 0}\frac{g'(\theta)}{g(\theta)}\sin\theta = 0$ is proved in (2.12.4). We can see

$$F(\pi) = \beta^2-\beta-\lim_{\theta\to\pi}\frac{g'(\theta)}{g(\theta)}\sin\theta(2-\alpha-2\beta) = \beta^2-\beta+2s(2-\alpha-2\beta) = (2s-\beta)(2s-\beta+1),$$

using (2.12.5)

$$\lim_{\theta\to\pi}\frac{g'(\theta)}{g(\theta)}\sin\theta = -2s.$$

Notice that we require $\gamma > 0$ small enough such that $\beta = \frac{2s}{2-\gamma} \le 1$ in the proof of (2.8.14). So $F(0) \le 0$ and $F(\pi) > 0$. Then we compute $F'(\theta)$:

$$\begin{aligned} F'(\theta) &= \beta(2-\alpha-2\beta)\sin 2\theta + \frac{1}{2}(2-\alpha-2\beta)\frac{gg''\sin 2\theta + 2gg'\cos 2\theta - (g')^2\sin 2\theta}{g^2} \\ &= \frac{\beta}{2}(2-\alpha-2\beta)(2-\alpha-\beta)\sin 2\theta + \frac{1}{2}(2-\alpha-2\beta)\frac{g'}{g}(-2+(2-2\alpha)\cos^2\theta) \\ &- \frac{1}{2}(2-\alpha-2\beta)(\frac{g'}{g})^2\sin 2\theta. \end{aligned}$$

When $F'(\theta) = 0$,

$$\sin 2\theta(\frac{g'}{g})^2 + (2-(2-2\alpha)\cos^2\theta)\frac{g'}{g} - \beta(2-\alpha-\beta)\sin 2\theta = 0.$$

Then

$$\begin{aligned} \frac{g'}{g} &= \frac{-(-1-(1-\alpha)\cos^2\theta) \pm \sqrt{(-1-(1-\alpha)\cos^2\theta)^2 + \beta(2-\alpha-\beta)\sin^2 2\theta}}{\sin 2\theta} \\ &= \frac{-(-1-(1-\alpha)\cos^2\theta) \pm \sqrt{L(\theta)}}{\sin 2\theta}, \end{aligned} \tag{2.12.8}$$

and let

$$L(\theta) = (-1-(1-\alpha)\cos^2\theta)^2 + \beta(2-\alpha-\beta)\sin^2 2\theta$$

By (2.12.7), we can compute that

$$\frac{g'}{g} = 2\frac{F(\theta) - (\beta^2-\beta)\cos^2\theta - \beta(2-\alpha-2\beta)\sin^2\theta}{(2-\alpha-2\beta)\sin 2\theta}. \tag{2.12.9}$$

Compare (2.12.8) and (2.12.9), we can compute that if $F'(\theta) = 0$ at some $\theta_0 \in (0,\pi)$, then at θ_0,

$$F(\theta) = (\beta^2-\beta)\cos^2\theta+\beta(2-\alpha-2\beta)\sin^2\theta+\frac{1}{2}(2-\alpha-2\beta)[-(-1-(1-\alpha)\cos^2\theta)\pm\sqrt{L(\theta)}]$$

is a bounded number. With the conditions that $F(0) = \beta^2 - \beta$ and $F(\pi) = (2s-\beta)(2s-\beta+1)$, we can prove that

$$|F(\theta)| \leq C(s,\gamma),$$

which is equivalent to

$$|\frac{U_{tt}}{U}| \leq \frac{C(s,\gamma)}{r^2}.$$

Now we prove (2.8.22)

$$|\frac{U_{tt}(\frac{\tau}{r},\frac{z}{r})}{U(\frac{\tau}{r},\frac{z}{r})}| \leq K_1.$$

Let $\theta = \arctan\frac{z}{\tau} \in [0,\pi]$, and we know $(\frac{\tau}{r},\frac{z}{r}) \in B^+_{3/2}/B^+_{1/2}$. Since U is homogeneous of degree β, we can see

$$|\frac{U_{tt}(\frac{\tau}{r},\frac{z}{r})}{U(\frac{\tau}{r},\frac{z}{r})}| = (\frac{r}{\sqrt{\tau^2+z^2}})^2|F(\theta)| \leq 4|F(\theta)|$$

with $F(\theta)$ defined in (2.12.7). Then using the results in the last part,

$$|\frac{U_{tt}(\frac{\tau}{r},\frac{z}{r})}{U(\frac{\tau}{r},\frac{z}{r})}| \leq 4|F(\theta)| \leq 4C(s,r) = K_1.$$

2.12.4 Proof of (2.8.23)

We prove if τ is between $t+\bar{t}$ and t, with

$$\bar{t} = -\frac{2(n-1)tr}{R} - \frac{\tilde{C}}{R^2}r^3 < 0,$$

then

$$U(\frac{\tau}{r},\frac{z}{r}) \le KU(\frac{t}{r},\frac{z}{r}).$$

Let $\theta_1 = arccos(\frac{\tau}{\sqrt{\tau^2+z^2}})$ and $\theta_2 = arccos(\frac{t}{r})$. Since $g(\theta) \ge 0$ and $g(\theta) = 0$ only when $\theta = \pi$, we only need to prove the inequality near $\theta_2 = \pi$. Since $(\frac{\tau}{r},\frac{z}{r}) \in B^+_{3/2}/B^+_{1/2}$, $t + \bar{t} \le \tau \le t$, and near $\theta_2 = \pi$, $t < 0$, we can see

$$0 < \pi - \theta_1 \le \pi - \theta_2.$$

As computed in (2.12.5),

$$\lim_{\theta\to\pi} \frac{g'(\theta)\sin\theta}{g(\theta)} = -2s < 0$$

with $g \ge 0$ and $\sin\theta \ge 0$, thus we can see

$$g'(\theta) \le 0$$

as $\theta \to \pi$. Therefore when θ_1, θ_2 are close to π

$$g(\theta_1) \le g(\theta_2),$$

and thus there exists $\bar{K} > 0$ such that

$$g(\theta_1) \le \bar{K} g(\theta_2)$$

for $\theta_1 = arccos(\frac{\tau}{\sqrt{\tau^2+z^2}})$ and $\theta_2 = arccos(\frac{t}{r})$. Therefore, there exists $K > 0$ such that

$$U(\frac{\tau}{r},\frac{z}{r}) \le (\frac{3}{2})^\beta g(\theta_1) \le (\frac{3}{2})^\beta \bar{K} g(\theta_2) = KU(\frac{t}{r},\frac{z}{r}).$$

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